This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

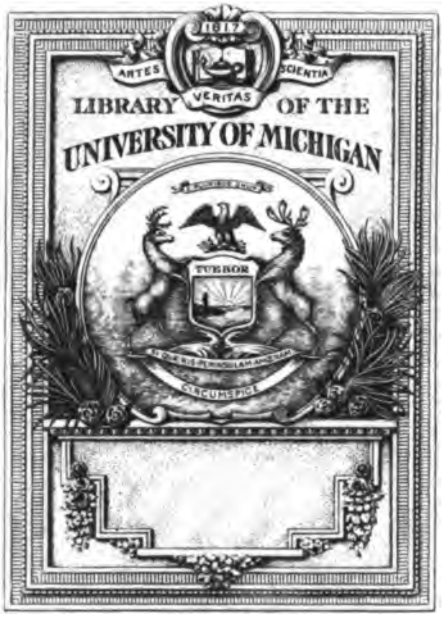
+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

B 4

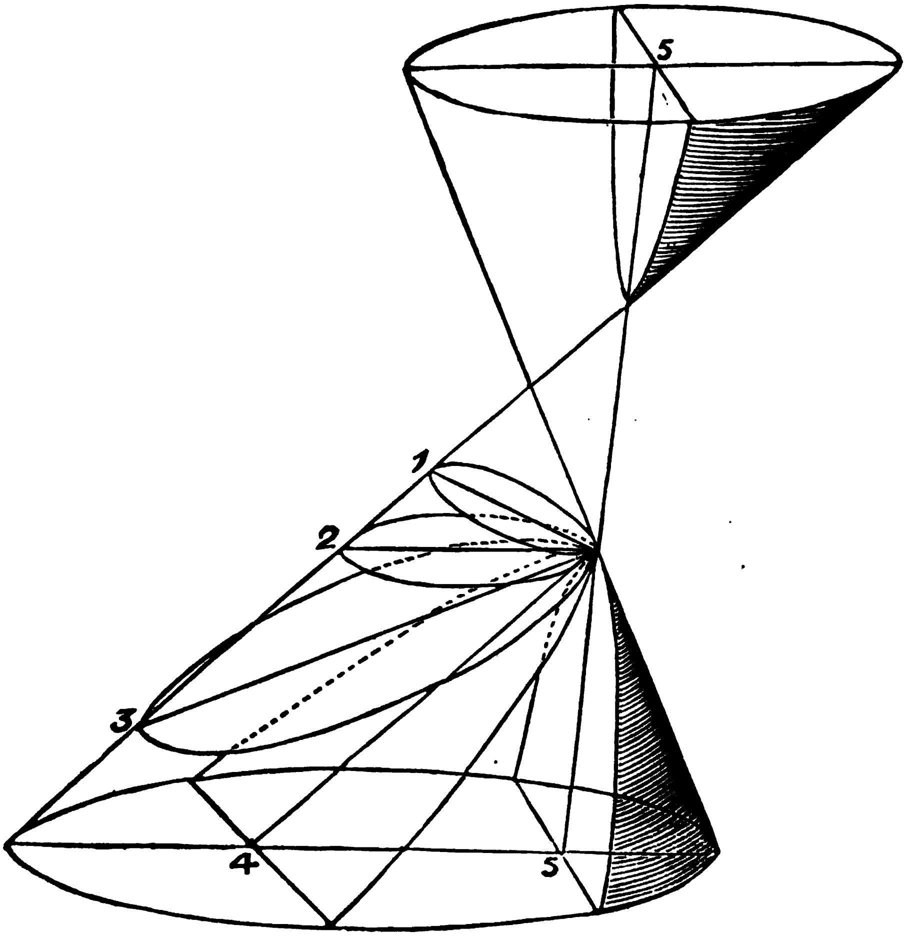


LIBRARY OF THE
UNIVERSITY OF MICHIGAN

RECEIVED IN EXCHANGE
FROM
Rhode Island
State College

QA
485
.R42

Samuel B. Wright, A. M., Ph. D.



“Καλείσθω δὲ ἡ τοιαύτη τομὴ

ΥΠΕΝΑΝΤΙΑ	1	Prop. V.
ΚΥΚΛΟΣ	2	„ IV.
ΕΛΛΕΙΨΙΣ	3	„ XIII.
ΠΑΡΑΒΟΛΗ	4	„ XI.
ΥΠΕΡΒΟΛΗ	5	„ XII.”

ΑΠΟΛΛΩΝΙΟΣ.

From the author

THE CONE AND ITS SECTIONS

TREATED GEOMETRICALLY,

BY

S. A. RENSHAW,

OF NOTTINGHAM.

ALL RIGHTS RESERVED.

London:

HAMILTON, ADAMS, & CO., PATERNOSTER ROW.

1875.

NOTTINGHAM:
PRINTED BY J. DERBY, ALBERT STREET.

Exch.
Librar.
R. I. State College
3-6-1941

PREFACE.

OF the works of the ancient Geometricians that have descended to us, none perhaps has been more justly celebrated than the Treatise of Apollonius, of Perga, on the Conic Sections—a work that has, apparently, maintained its superiority over every other Treatise that has since been written on the subject.

According to his Commentator, Eutocius, Apollonius was the first to derive the Sections from the Scalene Cone by different inclinations of the Cutting Plane, and in accordance with this derivation of the Curves he deduces the Primary Properties of the Sections from the Cone itself.

Modern writers have however to a considerable extent departed from this method—commencing their works by definitions of the Curves described in plano, such definitions being adapted to or derived from Properties of the Curves themselves; a strong tendency has however been shown, and especially by one Geometer of the last century, Hamilton, to revert to the Apollonian method. Accordingly in the present work the Primary Properties of the Sections are derived from the Cone, and it is believed that this is the case to a greater extent than in any previous Treatise. The reasons that have principally led to this course of treatment, are first, that the elementary Properties alluded to, and contained in Prop. V., are by the help of the most elementary Properties of the Cone demonstrated as it were together, and their connection and mutual dependence shown in the clearest manner; and secondly, it is presumed that by becoming acquainted with these principles as thus deduced, a deeper impression is obtained of their great and essential importance than could possibly be the case were that acquaintance made by deductions from any definition of the Curves in plano, seeing that according to the definition adopted the process of deduction must necessarily vary.

5-18-45 ADE

These Properties will then be found demonstrated both from the Scalene and Right Cone, and although their derivation from the former involves considerable prolixity, it is believed that the reader will be well repaid for the time and patience expended in the investigation.

The description of the Curves in plano and the derivation of their Properties thence, has however not been omitted; the method adopted in this part of the work being that of the Generating Circle, by which means not only the elementary, but many other Properties of the Sections can be proved both easily and elegantly. In the Articles on the Anharmonic Properties of the Sections, demonstrations of two important Theorems have been specially added in further illustration of the utility of this principle.

Of the general Theorems, one of the greatest importance and highly general application is that given as Prop. VII., and usually ascribed to Hamilton, although Apollonius had already given several cases of it. On this is founded another Theorem (VIII), by means of which the elementary relations of Diameters, their ordinates, and the tangents at their extremities are proved.

In the remaining parts on the seperate Sections, their areas, curvature, &c., the principal Theorems of the Curves will be found demonstrated by purely geometrical methods with but one exception in the case of the areas of Hyperbolic Trapezia, &c. In the Articles on Curvature, the plan adopted has been that of Wallace, though its identity with the method derivable from Prop. XI. is clearly shown; and in this as in other parts of the work, it is hoped that by a comparison of the different methods of treatment presented, a clearer and more complete conception of the principles of the Conic Sections will be obtained than could possibly be the case by the consideration of any one method alone.

The assistance derived from the works of other writers, both past and present, is gratefully acknowledged; amongst the latter may specially be mentioned those of Besant, Drew, Wilson, and Taylor.

In conclusion it is hoped that the reader will find this work of service in his prosecution of this highly useful and interesting branch of Mathematical Science, and that he will be urged to pursue it with vigour from the reflection, that these Curves, the Conic Sections, present us with no inconsiderable part of the actual design or plan, according to the principles of which, the Great Author of Nature has arranged some of the most stupendous of His glorious works, and that upon a knowledge of them, the intelligent comprehension of these sublime monuments and proofs of Himself and His attributes to a great extent depends.

S. A. RENSHAW.

Nottingham, Dec., 1874.

TABLE OF CONTENTS.

PAGE
Definitions 5
Elementary principles of the Cone—Propositions I., II., III., IV. 6
Prop. V. Primary principles of the Conic Sections derived from the Scalene Cone .. 9
Ditto from the Right Cone 14
Corollaries 1, 2, and 3 to Prop. 5. 15
Prop. VI. Demonstrating that the same Section may be cut from both a Scalene and 16
rallel lines cutting either a 17
rties of diameters and their 19
. .. . 23
n of the principle of, with Properties of the Section .. 26
. .. . 32
. .. . 34
. .. . 36
. .. . 38
. .. . 39
. .. . 48
. .. . 53
. .. . 66
Construction and Properties of the Hyperbola.. 71
The Rectangular Hyperbola 92
Examples on the Hyperbola and Rectangular Ditto 96
Similar Conics 104
Areas of the Conic Sections 106
Curvature of the Conic Sections 111
Anharmonic Ratio.. 120
Harmonic Ratio 124
Anharmonic Properties of Conics 128
Harmonic Ditto 134
Examples 141

ERRATA, &c.

Page 9, *lines* 31 and 34.—Ex æquali here refers to Eu. V. 11, and the same occurs in other places.

Page 13, *line* 4.—Read A′X.XA for AX′.XA.

Page 126, *line* 24.—Read 'Q and L' for 'P and L.'

Page 84, *line* 22.—Read 'Art. 126.'

Of the general Theorems, one of the greatest importance and highly general application is that given as Prop. VII., and usually ascribed to Hamilton, although Apollonius had already given several cases of it. On this is founded another Theorem (VIII), by means of which the elementary relations of Diameters, their ordinates, and the tangents at their extremities are proved.

In the remaining parts on the seperate Sections, their areas, curvature, &c., the principal
Theorems of the Curves will be found
exception in the case of the areas of H
the plan adopted has been that of Wal
Prop. XI. is clearly shown ; and in thi
comparison of the different methods
conception of the principles of the Cor
case by the consideration of any one m

The assistance derived from the w
acknowledged ; amongst the latter ma
and Taylor.

nclusion it is hoped that
useful and interestir
vigour from
ble part
ome of the most
intelligent comprehension of these sublime monuments and
ates to a great extent depends.

S. A. RENSHAW.

TABLE OF CONTENTS.

	PAGE
Definitions	5
Elementary principles of the Cone—Propositions I., II., III., IV.	6
Prop. V. Primary principles of the Conic Sections derived from the Scalene Cone	9
Ditto from the Right Cone	14
Corollaries 1, 2, and 3 to Prop. 5.	15
Prop. VI. Demonstrating that the same Section may be cut from both a Scalene and Right Cone, and by the same plane	16
Prop. VII. General Theorem relating to two pairs of parallel lines cutting either a Cone or Conic Section	17
Prop. VIII. General Theorem applied to establish the properties of diameters and their ordinates founded on the preceding Theorem	19
The Latus Rectum and Properties relating thereto	23
The Generating or Auxiliary Circle of a Conic, explanation of the principle of, with application of the same to establish several Primary Properties of the Section	26
Prop. IX. Theorem relating to an inscribed Trapezium	32
Prop. X. Theorem relating to angles subtended at foci	34
The Normal to a Conic	36
Prop. XI. Intersection of a Conic by a Circle	38
Construction of and Properties of the Parabola	39
Examples on Ditto	48
Construction of and Properties of the Ellipse	53
Examples on Ditto	66
Construction and Properties of the Hyperbola	71
The Rectangular Hyperbola	92
Examples on the Hyperbola and Rectangular Ditto	96
Similar Conics	104
Areas of the Conic Sections	106
Curvature of the Conic Sections	111
Anharmonic Ratio	120
Harmonic Ratio	124
Anharmonic Properties of Conics	128
Harmonic Ditto	134
Examples	141

THE CONE AND ITS SECTIONS.

DEFINITIONS.

Art. 1.

1.—If from any point, to the circumference of a circle, which is not in the same plane with the point, a right line be drawn, and produced both ways, and the point remaining fixed, the right line be turned about the circumference of the circle, until it return to the place from which it began to move. The Superficies described by the right line, and made up of two Superficies joined together at the Vertex, is called a CONICAL SUPERFICIES—and each of the two Superficies of which it is composed can be increased to infinity, by producing the right line by which it is described to infinity.

2.—The fixed point is called the Vertex.

3.—The right line drawn from the Vertex to the centre of the given circle is called the Axis.

4.—A Cone is the figure contained by the circle and that part of the Conical Superficies which lies between the Vertex and the circumference of the circle.

5.—The Vertex and Axis of the Cone are the same as those of the Conical Superficies, and the Circle is called the Base of the Cone.

6.—Cones which have their Axes at right angles to their Bases are called Right Cones, but those which have not are called Scalene or Oblique Cones.

7.—If a Conical Superficies, or which is the same thing, the two opposite Cones which it includes, be cut by a plane, the Section made thereby on the surface is called a Conic Section. If the cutting plane pass through the Vertex the Section is a Triangle, but if it be parallel to the Base or placed in a sub-contrary position the Section is a circle, as will be hereafter proved.

8.—If the plane cut the Conical Superficies, or one of the Cones, on the same side of the Vertex, and be neither parallel to the Base nor in a sub-contrary position, the Section is called an ELLIPSE. If it cut the Conical Superficies, or the two Cones, on opposite sides of the Vertex, the Section is called an HYPERBOLA. If it be parallel to a side of the Cone the Section is called a PARABOLA.

9.—If a plane drawn through the Axis and perpendicular to the Base of a Cone be at right angles to any other plane cutting the Cone, the line of intersection of the two planes is called the Axis of the Section formed by the latter.

10.—A plane is said to touch a Cone or Conical surface when it meets it in one position of the generating line only; and similarly a straight line is said to touch a Conic Section when it meets it in only one point.

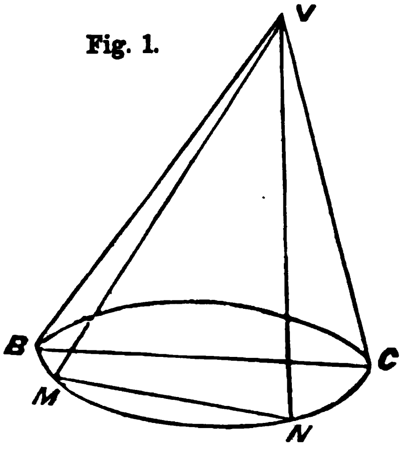
Art. 2.

PROP. I. THEOR.—*If a Cone be cut by a plane through the Vertex the Section will be a Triangle.*

Fig. 1. Let the Cone BVC be cut by the plane MVN, which passes through the Vertex. The Section MVN shall be a Triangle.

Let M and N be the points in which the plane cuts the circumference of the base, and join MN, then (by Eu. 3.2) MN is a straight line, and VM, VN are two positions of the generating line. Therefore MVN is a Triangle.

Fig. 1.



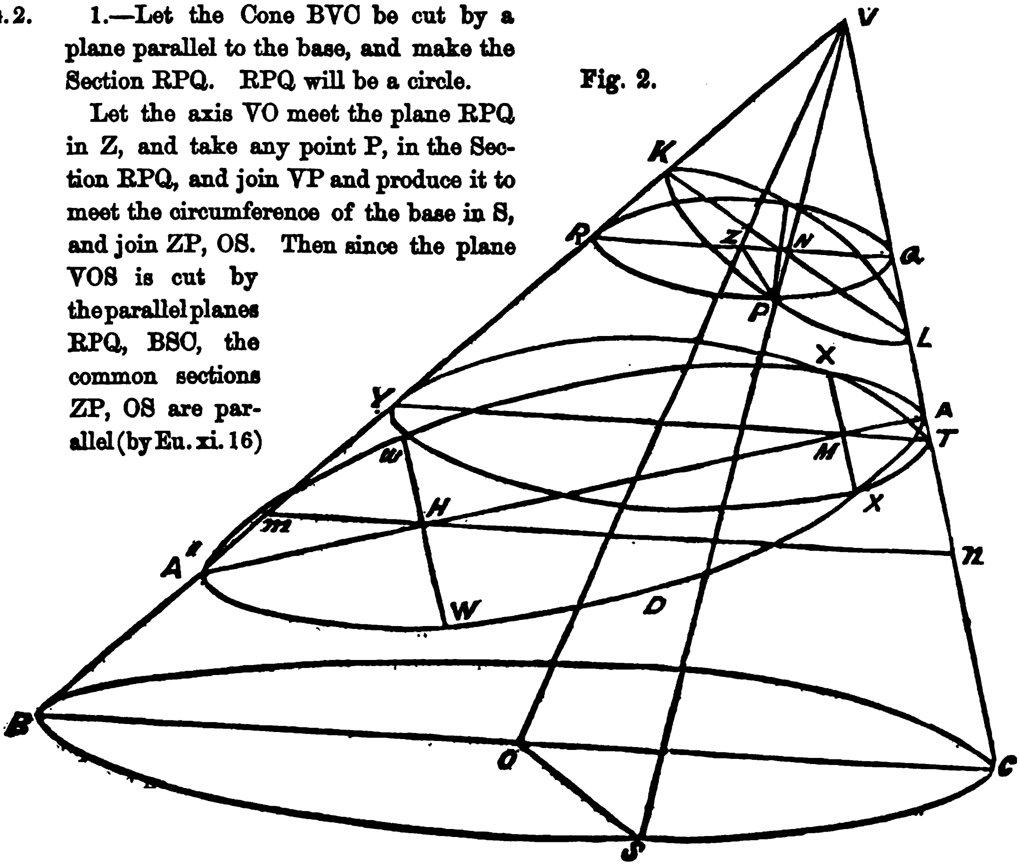
Art. 3.

PROP. II. THEOR.—*If a Cone be cut by a plane parallel to the Base or by a plane in a sub-contrary position, the Section will be a Circle.*

Fig. 2. 1.—Let the Cone BVC be cut by a plane parallel to the base, and make the Section RPQ. RPQ will be a circle.

Let the axis VO meet the plane RPQ in Z, and take any point P, in the Section RPQ, and join VP and produce it to meet the circumference of the base in S, and join ZP, OS. Then since the plane VOS is cut by the parallel planes RPQ, BSC, the common sections ZP, OS are parallel (by Eu. xi. 16)

Fig. 2.



For an exactly similar reason RQ is parallel to BC. Therefore by similar triangles

$$\begin{aligned} VO : VZ &= OB : ZR \\ &= OC : ZQ \\ &= OS : ZP. \end{aligned}$$

But OB, OC, OS are all equal. Therefore also ZR, ZQ, ZP are equal. Therefore the section RPQ is a circle of which the centre is Z. (Eu. iii. 9.)

Art. 4.

2.—From P draw PN at right angles to RQ, and through N draw KNL in the plane BVC so as to make the angle VLK equal to the angle VRQ, and let KPL be the section made by a plane passing through KL, NP. Then KPL, which is called a sub-contrary section, is a circle. For since the angle VLK is equal to the angle VRQ and the angles QNL, KNR being opposite are equal, the triangles QNL, KNR are similar. Therefore $RN : NK = LN : NQ$. Therefore $KN.NL = RN.NQ = NP^2$ by the circle RPQ. And since the same may be shewn for every point of the section KPL. Therefore the sub-contrary section KPL is a circle.

Art. 5.

Cor.—If ADA′ be a section* made by a plane, neither parallel to the base nor placed in a sub-contrary positon, but at right angles to the triangular section VBC that is perpendicular to the base of the Cone. And AA′ the axis of the section according to Def. 9. Then if any point X be taken in the section ADA′, and XM be let fall perpendicular to the triangular section VBC, and be produced to meet the Cone again in x, the line Xx will be bisected in the point where it meets the triangular section. For let it meet it in M, then (by Eu. xi. 38) the point M is in AA′. Through XMx draw a plane parallel to the base, which is possible since every plane passing through XMx is perpendicular to the plane VBC (by Eu. xi. 18) to which the base is also perpendicular. Consequently the section made by the plane through XMx will be a circle, let YMT be its Axis by Def. 9. Then Xx is at right angles to YT (Eu. xi. Def. 3) and is therefore an ordinate to the diameter of the circle YT, and is bisected in M. In like manner if any other line WHw be similarly drawn, in the section ADA′, to XMx, WH = wH. Through H draw mHn parallel to the base of the Cone, then by similar triangles

$$\begin{aligned} & A'H : A'M = mH : YM \\ \text{also} \quad & AH : AM = nH : TM \\ \therefore \quad & AH.HA' : AM.MA' = mH.Hn : YM.MT \\ & \qquad\qquad\qquad\quad = HW^2 : MX^2 \text{ by the circle.} \end{aligned}$$

Art. 6.

PROP. III. Prob.—*Having given a Scalene or Oblique or Right Cone, to draw a plane to touch it in a given position of the generating line*—

Fig. 3. Let BVC be the Cone, and VR the generating line in which it is to be touched by the plane.

Take O the centre of the base, and join OR, and in the plane BRC draw TRS at right angles to OR. Then the plane through VR, TRS is that which is required.

For take any point P (not in the line VR) in the plane VTS, and through P draw a plane parallel to the base of the Cone

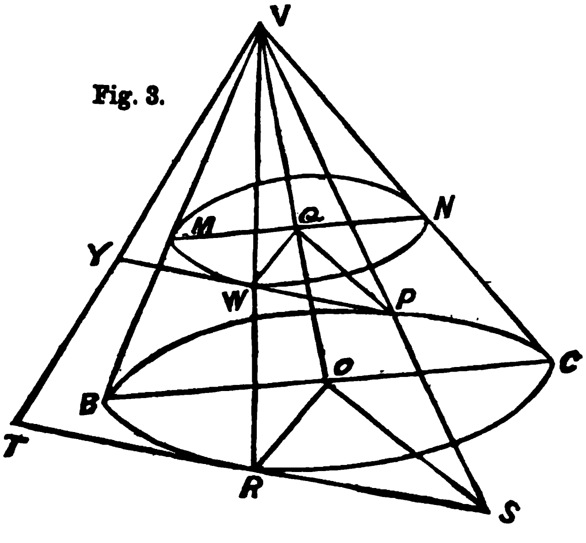


Fig. 3.

---

*The section here spoken of may either be the Parabola, Ellipse or Hyperbola.)

and let its section, which by **Prop.** II. is a circle, be MWN, and let the common section of this plane and the plane VTS be the straight line YWP; also let Q be the point in which the Axis VO meets the plane MWN. Join QW, QP, VP and produce the latter to meet TR in S. Join OR, OS.

Then (by Eu. xi. 16) WQ is parallel to OR and QP to OS. Therefore by similar triangles

$$VQ : VO = WQ : OR = QP : OS$$

But OR is less than OS because the point S falls without the circle BRC. Therefore also WQ is less than QP, and therefore the point P falls without the circle MWN, and therefore without the Cone BVC. And in the same way it may be shewn that any other point in the plane VTS, which is not in the line VR, falls without the Cone. But the line VR itself is in the surface of the Cone being one position of the generating line. Therefore the plane VTS meets the Cone in the line VR, but otherwise falls without the Cone. Therefore it is a Tangent Plane to the Cone in the line VR. which was to be drawn. (Def. 10.)

Art. 7.

PROP. IV. THEOR.—*If a Cone be cut by a plane, not passing through the Vertex, the common section of the cutting plane, with any Tangent plane to the Cone, touches the section made by the Cutting plane, in the point where the generating line, in which the Tangent plane touches the Cone, meets the section.*

(See Fig. to PROP. III.)

Let the Cone BVC be cut by any plane MWN, and touched by the Tangent plane TVS in the generating line VR, and let YWP be the intersection of the two planes, and MWN the section of the cone which meets the generating line VR in W. The right line YWP shall be a Tangent to the section MWN at the point W.

For the right line YWP is in the Tangent plane, and must therefore touch the Cone somewhere in the line VR. But YWP being also in the Cutting plane, must therefore touch the Cone in the point where the Cutting plane meets VR, and therefore it touches the section MWN in the point W. (Def. 10.)

Art. 8.

NOTE.—Propositions XVI., XXXVI., and XXXVII. of the 3rd Book of Euclid's Elements are equally true of the Sphere as of the Circle, and may be demonstrated in the same way. In what follows they will be quoted as the XVI., &c., of the 3rd Book, and as applying to the Sphere.

Art. 9.

PROP. V. THEOR.—*In the Ellipse and Hyperbola the Tangent at any point on the curve makes equal angles with the lines drawn from the point of contact to the Foci. But in the Parabola it bisects the angle contained by a line drawn from the point of contact to the Focus, and the perpendicular from the same point on the Directrix.*

Also in the Ellipse the sum and in the Hyperbola the difference of the Focal distances of any point on the curve is equal to the Axis of the curve. But in the Parabola the distance of any point on the curve from the Focus is equal to the perpendicular from the same point on the Directrix.

FIGS. 4 & 5. Let BVC be a scalene or oblique Cone, and let it be cut by a plane so as to make the Elliptic section (in Fig. 4) or the Hyperbolic section (in Fig. 5.) And let AA′ be the common section of the cutting plane and of the triangular section through the Axis, and which is at right angles to the cutting plane and also to the Base, and consequently AA′ is the Axis of the section. (Def. 9.)

In the triangular section draw AD, A′E parallel to the base of the Cone, and bisect them in H and H′. Divide AA′ in F and F′, so that AF, FA′ = AF′, F′A′ = AH, A′H′. Then F and F′ are the foci of the curves.

For let P be any point on the curve, and BVC the triangular section through the Axis at right angles to the base. Draw the generating line VP and produce it both ways. Join FH, F′H′ and produce them to meet BV, VC, in K and K′. Draw KLX, K′L′X′ parallel to AH or A′H′ and meeting AA′ in X and X′.

Then since $\text{AF.FA}' = \text{AH.A}'\text{H}'$

$\text{A}'\text{F} : \text{A}'\text{H}' = \text{AH} : \text{AF} = \text{KX} : \text{FX}$ by similar triangles.

But since AD is bisected in H and is parallel to KX, therefore KA′, KF, KA, KX form a harmonic pencil, and therefore AA′ is harmonically divided in the points F and X.

Therefore $\text{A}'\text{F} : \text{A}'\text{X} = \text{AF} : \text{AX}$

Comp$^{\text{do}}$ $2\text{A}'\text{F} + \text{FX} : \text{A}'\text{X} = \text{FX} : \text{AX}$

Alt$^{\text{do}}$ et div$^{\text{do}}$ $2\text{A}'\text{F} : \text{FX} = \text{AA}' : \text{AX}$

But by sim tri $\text{AA}' : \text{AX} = 2\text{A}'\text{H}' : \text{LX}$

∴ Ex æquali $2\text{A}'\text{F} : \text{FX} = 2\text{A}'\text{H}' : \text{LX}$

Alt$^{\text{do}}$ $\text{A}'\text{F} : \text{A}'\text{H}' = \text{FX} : \text{LX}$

But it has been proved above that $\text{A}'\text{F} : \text{A}'\text{H}' = \text{KX} : \text{FX}$

Therefore ex æquali $\text{FX} : \text{LX} = \text{KX} : \text{FX}$

Therefore $\text{FX}^2 = \text{LX.KX}.$

Again, from X draw XYW, meeting VC, VB in Y and W respectively, and so as to

---

Art. 10.

NOTE.—To divide AA′ so that AF.FA′ = AF′.F′A′ = AH.A′H′. On AA′ describe a semi-circle with centre O. Take the chord Am equal to the sum or difference of AH, A′H′ according as it is for the Ellipse or Hyperbola; and take An equal to the greater of the two AH, A′H′. Join On. The circle described from the centre O with radius On, will cut AA′ in the required points F and F′.

make the angle XYL equal to the angle WKX, then the triangles WKX, YLX being similar $KX : XW = YX : XL$. Therefore $KX.XL = WX.XY = FX^2$.

Draw Ad, A′e, so as to make the same angles with the side VC that AD does with VB; also, draw EG making the angle A′EG with A′E, equal to the angle DA′A, and let it meet A′A in G, then by similar triangles $LA : AX = AE : AA'$

$$\text{and } YA : AX = Ae : AA'$$

$$\text{Therefore} \qquad LA.YA = \frac{AX^2,\ AE.Ae}{A'A^2}$$

$$\text{But since } A'X : AX = A'F : FA$$

$$\text{Div}^{\text{do}} \quad AA' : AX = 2FO : FA, \text{ O being the middle of } AA'$$

$$\text{Therefore} \qquad \frac{AX}{AA'} = \frac{FA}{2FO}$$

$$\text{Therefore} \qquad LA.YA = \frac{FA^2,\ AE.Ae}{4FO^2}$$

$$\text{But } OF^2 = AO^2 - 2AF.FO - AF^2 = AO^2 - AF.FA' = AO^2 - AH.A'H'$$

$$\text{and } 4OF^2 = 4AO^2 - 4.AH.A'H' = A'A^2 - AD.AE$$

Also, by the similar triangles A′EG, AA′D,

$$AA' : AD = A'E : A'G \therefore AD.A'E = AA'.A'G;$$

$$\text{Therefore} \qquad LA.YA = \frac{FA^2.AE.Ae}{A'A^2 - A'A.A'G} = \frac{FA^2.AE.Ae}{A'A.AG.}$$

But since the angle $A'GE = A'DA$, therefore the remaining angles AGE, VDA, are equal and $VDA = VeA'$, therefore the triangles GAE, A′Ae, are similar,

$$\text{And } Ae : AA' = AG : AE \therefore Ae.AE = AA'.AG,$$

$$\text{Therefore } LA.YA = FA^2.$$

Hence if FL, FY be joined, the angle $LFA = AYF$,

But $LFA = LKF$. (Since $LX.XK = FX^2$.)

Therefore $LKF = LYF$.

Hence a circle may be described through the five points W, Y, L, F, K, let this circle be described and let the sphere of which it is a great circle be completed, and let the centre of the sphere be R, which may be found by drawing FR at right angles to the plane APA′, and QR at right angles to KL from its middle point, to intersect in R.

Let the Sphere and the Cone be cut by planes through WYX, KLX, and at right angles to the plane BVC. Through X draw XT, in the plane of the Ellipse or Hyperbola, at right angles to A′AX; and let the tangent plane to the Cone through VP, cut XT in T, and join TN, TM (N and M being the intersections of the planes through WYX, KLX, with the generating line VP.)

Then TN, TM, are tangents to the circular sections WNY, KML, (by Prop. IV.) and they are also tangents to the sphere. For if RN, RM, be joined, then RN, RM, are at right angles to NT, MT. Join QM, QT, then $RT^2 = QT^2 + RQ^2 = TM^2 + QM^2 + RQ^2 = TM^2 + RM^2$. Therefore (by Eu. I. 48) RM is at right angles to

See Note (A), page 15.

MT, and in the same manner it may be shewn that RN is at right angles to NT; but RM, RN, are radii of the Sphere, therefore (by Eu. III. 16) TM, TN, are tangents to the Sphere, and TF evidently is, since RF is at right angles to the cutting plane. (Eu. XI. Def. 3.)

Therefore TF, TM, TN, being tangents to the Sphere from the same point T are equal to one another; and in the same way if the same construction be made with regard to the other focus F′ and the lower sphere whose centre is R′, it may be proved that T′F′, T′M′, T′N′ are also all equal to one another; and further, TM, TN, T′M′, T′N′ being all in the same tangent plane, viz., that through VP, TM is parallel to T′M′ and TN to T′N′, therefore by similar triangles

$$PT : PM = PT' : PM'$$
$$PT : PN = PT' : PN'.$$

Compounding and alternately $PT^2 : PT'^2 = PM.PN : PM'.PN'$
$= PF^2 : PF'^2$ (Eu. III. 36.)

Again, since MTN is an isosceles triangle, for TM = TN, therefore $TP^2 = TM^2 + NP.PM = TM^2 + PF^2 = TF^2 + FP^2$ by what has been proved and Eu. III, 36. Therefore TFP is a right angle. Similarly T′F′P may be shown to be a right angle; and it has been proved above that $PT : PT' = PF : PF'$.

Therefore (by Eu. VI. 7) the triangles TFP, T′F′P are similar, and the angle TPF is equal to the angle T′PF′. That is, the lines drawn from the point of contact P, to the foci, make equal angles with the tangent.

Cor. 1.—From the above demonstration it is evident that the points L, F, H′, are in one right line, as are also the points L′, F′, H.

Cor. 2.—From the above it is also evident that

$$4OF^2 = FF'^2 = AE.Ae = A'D.A'd \text{ by sim. tri.}$$

Def.—The lines XT, X′T′, are called the Directrices of the Ellipse and Hyperbola.

Art. 11.

Fig. 6.

Next let APA′ (Fig. 6) be the section of a scalene or oblique Cone BVC, parallel to the side VB, and at right angles to the triangular section BVC through the axis of the Cone and perpendicular to its base. Consequently APA′ is a Parabola (Def. 8) and AA′ its axis. (Def. 9.)

Draw AD parallel to the base of the Cone and meeting VB in D. Bisect AD in H, and AH in I, and draw IF so as to make the angle AIF equal to the angle A′AC, and let it meet AA′ in F. Then F is the focus of the Parabola.

Through F draw ZFG parallel to the base of the Cone, join FH and produce it to meet VB in K, draw KLX parallel to the base of the Cone, and complete the scheme as in the cases of the Ellipse and Hyperbola.

Then exactly as was shown in those cases the angle PFT may be shown to be a right angle. Draw DQ parallel to IF, and meeting AA′ in Q, also draw PN at right angles to AA′ and mNn parallel to the base of the Cone.

Then by similar triangles $DA : AQ = AN : Nn$

Therefore $DA.Nn = AQ.AN$; that is $4AF.AN = NP^2$ or $4AF^2 + 4AF.FN = NP^2$

Therefore adding to both sides FN^2 we have

$$4AF^2 + 4AF.FN + FN^2 = NP^2 + FN^2$$

That is, since $FA = DK = AX$; $NX^2 = FP^2$

Therefore $FP = NX = PM$.

Again, since by similar triangles $TP : PM (= PF) = PE : EN$

And the angles PFT, PNE, are right; therefore (by Eu. VI. 7) the triangles FPT, PEN are similar, and the angle FPT equals the angle PEN, but PEN equals EPM, because they are alternate. Therefore the angle FPE equals the angle EPM.

Therefore the Tangent bisects the angle contained by the line drawn from the point of contact to the Focus, and the perpendicular from the same point on the line XT.

Def.—The line XT is called the Directrix of the Parabola.

This completes the proof of the Theorem for the Parabola derived from the Oblique Cone. In the next article the Theorem will also be completed for the Ellipse and Hyperbola.

Art. 12.

Figs. 7 & 8

To prove the second part of the Theorem for the Ellipse and Hyperbola, join FP, F′P (see Figs. 7 and 8) and on AA′ describe a circle in the plane of the section, and let O be its centre. Draw FR (Fig. 7) or XR (Fig 8) at right angles to AA′ and let it meet the circle thereon in R. Join XR (Fig. 7) or FR (Fig. 8) which will be tangents to the circle since AA′ is harmonically divided in F and X. Draw PN at right angles to AA′, and PP′ parallel to AA′ and meeting the section again in P′, also draw P′N′ at right angles to AA′. Join RO and draw Nn, N′n′, perpendicular thereto.

Then since $PN = P'N'$, ON will equal ON′, for if through N and N′ parallels be drawn to the base of the Cone, by compounding the ratios of the sides of similar triangles, the rectangles AN.NA′, AN′.N′A′ are to one another as the rectangles of the segments of the parallels through N and N′, which latter rectangles are equal to NP^2 and $N'P'^2$ (by Prop. II.) But $NP = N'P'$ therefore $AN.NA' = AN'.N'A'$, that is $AO^2 - ON^2 = AO^2 - ON'^2$, therefore $ON^2 = ON'^2$ and $ON = ON'$.

Hence, since $NP = N'P'$ and $NF = N'F'$ therefore (by Eu. I. 4) FP' will equal $F'P$

Then in the Ellipse (Fig. 7) by drawing through N and N′ parallels to the base of the Cone and compounding the ratios of the sides of similar triangles

$$AX'.XA : KX.XL = A'N'.N'A : N'P'^2$$

That is $XR^2 : FX^2 = A'O^2 - ON^2 : N'P'^2$

But by sim. tri. $XR : FX = N'O : N'n'$

∴ Ex æquali $A'O^2 - ON'^2 : N'P'^2 = N'O^2 : N'n'^2$

Alter$_{do}$ et comp^{do} et alt^{do} $A'O^2 - ON'^2 : N'P'^2 = AO^2 : N'P'^2 + N'n'^2$

But from above $AO^2 - ON'^2 : N'P'^2 = RX^2 : XF^2$

And by sim. tri. $RX^2 : XF^2 = OR^2\ (= OA^2) : FR^2$

∴ Ex æquali $OA^2 : FR^2 = OA^2 : N'P'^2 + N'n'^2$

Therefore $FR^2 = N'P'^2 + N'n'^2$

Add to each FN'^2 and $FR^2 + FN'^2 = FN'^2 + N'P'^2 + N'n'^2$

That is $N'R^2 = FP'^2 + N'n'^2$

But $N'R^2 = Rn'^2 + N'n'^2 = FP'^2 + N'n'^2$

Therefore $Rn'^2 = FP'^2$ and $Rn' = FP'$

Similarly $Rn = FP$

From A draw the tangent AZ to the circle KFL and produce it towards A. Join XZ and draw NG, OM, N′Q parallel thereto, and meeting AZ in the points G, M, Q.

Then since AA′ is harmonically divided in F and X

$$XA : AF\ (= AZ) = XR : RF$$

And by sim. tri. $XA : AZ = XN : ZG$

Also $XR : RF = XN : Rn$

Therefore Ex æquali $XN : ZG = XN : Rn$

Therefore $ZG = Rn$

Similarly $ZQ = Rn'$ and $ZM = RO = OA$

But since $ON = ON'$ therefore $ZG + ZQ = 2ZM = 2OA$

Therefore $FP + F'P = Rn + Rn' = ZG + ZQ = 2OA = AA'$

Art. 13.

FIG. 8. Making the same construction for the Hyperbola as for the Ellipse, and compounding the ratios of the sides of similar triangles. (Fig. 8.)

$$A'X.AX : XK.XL = A'N'.N'A : P'N'^2$$

That is $RX^2 : XF^2 = ON'^2 - OA^2 : P'N'^2$

But $RX : XF = OR\ (= OA) : RF$ by sim. tri.

∴ Ex æquali $ON'^2 - OA^2 : P'N'^2 = OA^2 : RF^2$

Alt^{do} comp^{do} et alt^{do} $ON'^2 - OA^2 : PN'^2 = ON'^2 : PN'^2 + RF^2$

But by above $ON'^2 - OA^2 : PN'^2 = RX^2 : XF^2$

$= ON'^2 : N'n'^2$ by sim. tri.

Therefore ex æquali $ON'^2 : N'n'^2 = ON'^2 : PN'^2 + RF^2$

Therefore $N'n'^2 = PN'^2 + RF^2$

To each add FN'^2 and $N'n'^2 + N'F^2 = N'F^2 + PN'^2 + RF^2$

That is $n'F^2 = PF^2 + RF^2$

But $n'F^2 = n'R^2 + RF^2 = PF^2 + RF^2$

Therefore $n'R^2 = PF^2$ and $n'R = PF$

Similarly $Rn = PF$

And exactly in the same way as in the Ellipse it may be shown that

$$ZG = Rn, \quad ZQ = Rn', \quad \text{and } ZM = RO = OA$$

Therefore $ZQ - ZG = 2ZM = 2OA$

And $FP - PF = Rn' - Rn = ZQ - ZG = 2OA = AA'$.

Therefore in the Ellipse the sum, and in the Hyperbola the difference of the Focal distance of any point on the curve is equal to the Axis.

Art. 14.

The properties enunciated in this Proposition (V) are easily proved in sections derived from right Cones as follows:—

FIGS. 9 & 10. Let BVC be a right Cone, and let it be cut by a plane so as to make the Elliptic section in (Fig. 9) and the Hyperbolic in (Fig. 10.) Inscribe spheres so as to touch the sides VB, VC, AA' in the points K, L, F; K', L', F'. Then F, F' are the foci of the curves.

Let R, R' be the centres of the inscribed spheres. Join KL, K'L', and produce them to meet AA' or that produced in X, X'. Take any point P in the section, and join VP, and through VP draw a tangent plane to the Cone (by Prop. 3) and let its intersection with the plane of the section be PTT', which (by Prop. 4) will be a tangent to the section. Draw XT, X'T' in the plane of the section and perpendicular to AA', and meeting PTT' in T and T'. And let MT, M'T' be the intersections of the tangent plane through VP to the Cone, with the planes through KLX, K'L'X' parallel to the base of the Cone. Therefore (by Prop. 4) MT, M'T' will be tangents to the circles of contact KLM, K'L'M'. Join RF, R'F', RM, R'M', RT, R'T', and complete the scheme.

Then since RFP, RMP are right angles

$$RM^2 + MP^2 = RP^2 = RF^2 + FP^2$$

But $RM = RF$. Therefore $PM = PF$. Similarly $PM' = PF'$

Therefore $PF \pm PF' = MM' = LK' = AA'$

The + or — sign being taken according as the section is an Ellipse or Hyperbola.

Again,—Since RMT, RFT are right angles

$$RM^2 + MT^2 = RT^2 = RF^2 + FT^2$$

But $RM = RF$. Therefore $MT = TF$. And similarly $M'T' = T'F'$.

Also since the plane KML and the tangent plane are both at right angles to the plane VRM, therefore (by Eu. XI. 19) MT is at right angles to the plane VMP and to the line VP; that is, PMT is a right angle. Similarly PM'T' is a right angle. Therefore $PT^2 = PM^2 + MT^2 = PF^2 + FT^2$ by what has just been proved. Therefore (by Eu. I. 48) PFT is a right angle, and in the same manner PF'T' may be shown to be a right angle; and since MT, M'T' are parallels

By similar triangles $PT : PM = PT' : PM'$

That is $PT : PF = PT' : PF'$

And the angles PFT, PF'T' have been shown to be right angles. Therefore (by Eu. VI. 7) the triangles PFT, PF'T' are similar and the angle FPT equals the angle F'PT'. That is the tangent makes equal angles with the lines drawn from the point of contact to the Foci.

Art. 15.

Cor. 1.—If through P, PNN' be drawn at right angles to the lines XT, X'T', that is to the directrices of the curves and meeting them in N and N', then

By similar triangles $PT : PT' = PN : PN'$

But $PT : PT' = PF : PF'$

∴ Ex æquali $PF : PF' = PN : PN'$

Comp^{do} or div^{do} $PF : PF \pm PF' = PN : PN \pm PN'$

That is alternately $PF : PN = AA' : XX'$

Similarly $PF' : PN' = AA' : XX'$

Hence the distance of any point on the curve from the focus is to its distance from the corresponding directrix in the constant ratio of AA' to XX'

It is also to be observed that since A and A' are points on the curve that

$$AF : AX = A'F : A'X = AA' : XX'$$

This ratio of AF to AX, or what is the same thing, the ratio of the distance of any point on the section from the focus, to its distance from the Directrix, is called the Eccentricity of the section.

Cor .2—If O be the middle point of AA, then since AA' is harmonically divided in F and X, $XO.OF = OA^2$.

Cor. 3.—In the course of the proof it has been shewn that if the tangent meet the directrices in T and T', and FT F'T' be joined, that these lines are at right angles to FP, F'P, respectively; and therefore if a focal chord FP be produced to meet the section again in p, the tangents at P and p meet in the same point T on the directrix.

A.—The first part of the demonstration of Prop. V. from the Oblique Cone, is adapted to the Ellipse. The alterations for the Hyperbola being as follows: Alt^{do} et comp^{do} in the proportion $2A'F : FX = AA' : AX$ on page 9; and comp^{do} in the proportion $AA' : AX = 2FO : FA$ on page 10; also, $OF^2 = AO^2 + 2AF.FO - AF^2 = AO^2 + AF.FA' = AO^2 + AH.A'H'$, and $4OF^2 = 4AO^2 + 4AH.A'H' = A'A^2 + AD.A'E = A'A^2 + AA'.A'G$.

Art. 16.

PROP. VI. Prob.—*Having given an Oblique Cone, and a Section made thereon by a cutting plane, to find the right Cone on which the same section is produced by the same cutting plane.*

Figs. 11, 12, & 13. Let BVC be an oblique Cone, and APA′ the section made on it by a cutting plane. (Figs. 11, 12, and 13.) Let AA′ be the axis of the section. Find the foci F and F′ as in Prop. V, and describe a circle FLK as in that Proposition. Draw the tangents Al, A′k to the circle FLK, and let them produced meet in Q. Let R be the centre of the circle FLK, and join kl, QR. Then the Cone of which Q is the vertex, QR (produced) the axis, and the base of which is parallel to kl is the right Cone required; that is, it will be intersected in the same section APA′ as was made on the oblique Cone and by the same plane.

Through A and A′ draw Ad, Ae, parallel to kl. Join Fk meeting Ad in h, then Ah is equal to dh, for draw dn parallel to AA′, and meeting Fk in n. Then since $A'F = A'k$, $dn = dk$; but $dk = lA = AF$. Therefore $AF = dn$, and the angles AhF, hAF, are equal to the angles nhd, hdn. Therefore (by Eu. I. 26) $dh = hA$. Therefore kA′, kF, kA, kl, form a harmonic pencil, and therefore kl must pass through X.

By Eu. III. 36 $\qquad kX.Xl = KX.XL$
Therefore $\qquad KX : kX = lX : XL$
But by sim. tri. $\qquad KX : kX = AD : Ad$
Also $\qquad lX : LX = A'e : A'E$
Therefore ex æquali $\qquad AD : dA = A'e : A'E$
Therefore $\qquad AD.A'E = Ad.A'e.$

Now if any point P in the section be taken and PG be drawn perpendicular to AA′, and through G a parallel to the base of each Cone be drawn, we have by compounding the ratios of the sides of similar triangles

In the oblique Cone $\qquad AA'^2 : AD.AE = AG.GA' : GP^2$
In the right Cone $\qquad AA'^2 : Ad.A'e = AG.GA' : GP^2$

Therefore since $AD.AE = Ad.A'e$ as has been proved, AG.GA has to GP^2 the same ratio whichever Cone GP be taken in; that is, the point P is common to the two Cones. In the same manner it may be shown that every point of the section is common to the two Cones. The section is therefore the line of intersection of the oblique and right Cones, and may therefore be equally derived from either by the same cutting plane. For it is evident that if QR be produced to any point Y, and MYN be drawn parallel to kl and meeting QA, QA′ in M and N, and a circle be described with radius YM or YN, the Cone having the vertex Q, and the base MN, will be right angled, and will be one from which it has been shown that the section can be equally derived from with the oblique Cone.

[*] For completing the sphere of which kFl is a great circle, let kfl be the circle in which tangents drawn from the point Q meet the sphere, and take any point f in this circle. Join Qf and produce it to meet the plane of the circle MN in W. Join YW, then i being the centre of the circle kfl

By similar triangles $Qi : if = QY : YW$
And $Qi : il = QY : YN$
∴ Ex æquali $if : il = YW : YN$
But $if = il$. Therefore $YW = YN$.

Therefore the point W falls on the circumference of the circle of which the centre is Y, and radius YM

Therefore YQN is the right angled Cone that was required to be drawn.

Art. 17.

Note.—In Prop. V. the properties of the sections contained in the enunciation are proved separately, both from the oblique and right Cone; and in Prop. VI it is shown that any section derived from an oblique Cone may also be derived by the same cutting plane from a right Cone which is found. It would therefore have been sufficient to have demonstrated the properties of Prop. V. from the right Cone alone; but on account of the importance of the properties themselves, and being under the impression that the Student will find the demonstration from the oblique Cone interesting and of value, they are given fully.

Art. 18.

PROP. VII. Theor.—*If from any two points either within or without a Cone or Conical Superficies, two pairs of parallel lines be drawn, meeting the Cone or Superficies, the ratio of the Rectangles under the Segments of one pair of parallels will be the same as the ratio of the Rectangles under the Segments of the other pair.*

Fig. 14. Let P and Q be any two points either within or without the Cone BVC, and from P and Q let the parallels PRS QMN[†] be drawn meeting the Cone in the points R,S, M,N from the Vertex V, draw VK parallel to MN, and meeting the base of the Cone in K, and let BCK be the intersection of the plane through VK, QMN with the base, and let Qmn be the intersection of the same plane, with the circular section through Q parallel to the base (Prop. II.) Then since VK is parallel to QMN, it is also parallel to PRS

---

[*] See Fig. 11.

[†] Note.—The diagram (14) exhibits two cases of the Theorem, viz., when both the lines PRS, QMN meet the same Cone, or when R′PS′, N′QM′ both meet opposite Cones.

(by Eu. XI. 9). Let a plane pass through VK and PRS and let EFK be its intersection with the base, and Prs that with the circular section through P. Then by similar triangles

$$NQ : nQ = VK : KC$$
$$QM : Qm = VK : KB$$

Compounding $NQ.QM : nQ.Qm = VK^2 : BK.KC$

Similarly $SP.PR : sP.Pr = VK^2 : EK.KF \ (= BK.KC)$

Therefore $NQ.QM : nQ.Qm = SP.PR : sP.Pr$

Alternately $NQ.QM : SP.PR = nQ.Qm : sP.Pr$

But $nQ.Qm : sP.Pr$ is a constant ratio, and is the same as that of the rectangles under the segments of any other pair of parallels drawn through P and Q to meet the Cone. Therefore if from any two points, &c.

Art. 19.

Cor.—When one or both the points are without the Cone, and any of the lines drawn to the Cone touch the same, then the rectangles under the segments become the squares of the said line or lines. Also when either or both points are within the Cone, and any of the lines drawn through them are bisected by such point or points, in this case the rectangles under the segments of the bisected lines become the squares of half the lines.

Art. 20.

Figs. 15 & 16. To apply this Theorem to the Sections of the Cone, take the points P and Q in the same plane that is in the plane of the Section considered.

For instance (see Figs. 15 and 16) when the lines drawn through P and Q all meet the Section, we have as stated in the proposition $NQ.QM : SP.PR = nQ.Qm : sP.Pr$

When any of the lines is bisected at the point through which it is drawn or touches the curve what is stated in the Corollary must be observed thus (Figs. 15 and 16). Suppose P′ and Q′ to be the points through which the parallels P′sr, Q′m′n′—and also P′R′, Q′MN are drawn, and suppose MN is bisected in Q′ and P′R′ is a Tangent. Then

$$NQ'^2 : P'R'^2 = n'Q'.Q'm' : sP'.P'r$$

and in like manner in other similar cases.

Art. 21.

Fig. 17. As a special case of this Theorem, the following remains to be noticed:—When two parallels as SR, MN are cut by a single straight line as QPL which itself is parallel to a side of the Cone and only meets the section in one point, and when this is the case the section will be either a Parabola or Hyperbola, the Theorem becomes

$$SP.PR : NQ.QM = PL : LQ$$

For let the plane passing through SR and the Vertex cut the base in the line KFE and draw VK parallel to SR and complete the figure. Then

By sim. tri. $SP : Ps = VK : KE$

And $RP : Pr = VK : KF$

Compounding $SP.PR : sP.Pr = VK^2 : EK.KF$

But $SP.Pr = HP.PI$ by the circle

Therefore $SP.PR : HP.PI = VK^2 : EK.KF$

Similarly $NQ.QM : hQ.Qi = VK^2 : EK.KF$

Therefore $SP.PR : NQ.QM = HP.PI : hQ.Qi$

Since $HP = hQ$ $\qquad = PI : Qi$

By sim. tri. $\qquad = PL : QL$

as above stated.

Art. 22.

PROP. VIII. THEOR.—*The centres of all straight lines drawn in a Conic Section parallel to any Tangent thereto lie on a straight line passing through the point of contact.*

FIGS. 18 & 19 Let AA′ be a Conic section, and CAH any tangent thereto at A, the centres of all straight lines drawn in the section parallel to CAH will lie on a right line passing through A

For let DE be any chord of the section parrallel to CAH and N the middle point of DE. Join AN, and take any point K in the section. Join DK and produce it to meet NA produced in T. Join TE meeting the section again in G, and join KG. Then KG is parallel to CAH and is bisected by AN.

Let DT, ET meet CAH in the points C and H, and through N draw XNX′, YNY′ parallel to DT and ET respectively.

Then by Prop. VII. and since $AC = CH$.

$$DT.TK : ET.TG = XN.NX' : YN.NY' \qquad (1)$$

Also $DC.CK : CA^2 = XN.NX' : DN.NE$

And $CA^2(= CH^2) : EH.HG = DN.NE : YN.NY'$

∴ Ex æquali $DC.CK : EH.HG = XN.NX' : YN.NY'$

Therefore by (1) $DT.TK : ET.TG = DC.CK : EH.HG$

That is $(DC + CT)(KC + CT) : (EH + HT)(GH + HT) = DC.CK : EH.HG.$

Alt^{do} div^{do} et alt^{do} $(DT + KC)\,CT : (ET + GH)\,HT = DC.CK : EH.HG$

But by parallels DE, CH $CT : HT = DC : EH$

Therefore $DT + KC : ET + GH = CK : HG$

Alt^{do} and div^{do} $DT : ET = CK : HG$

Therefore KG is parallel to DE or CH. But AN bisects DE, therefore it also bisects KG. And since K is any point in the section, it is evident that the centres of all chords parallel to CAH lie on a right line passing through A.

And it will be observed that if in the Ellipse the tangent parallel to DE had been on the other side of that line and towards opposite parts of the curve as C′A′H′, or if in the Hyperbola the tangent parallel to DE had touched the opposite branch as C′A′H′, it can still be proved by an exactly similar process that KG (or K″G″) is parallel to DE and bisected by AN or A′N. The proposition is therefore proved.

FIG. 18. The converse of Prop VIII is also true, viz. If DE and KG be any two parallel chords of a conic section, and be bisected in N and I, and NI be joined and produced both ways to meet the section in A and A′, the lines drawn through A and A′ parallel to DE or KG shall touch the section. For if not let CAH (parallel to DE or KG) meet the curve again in the point V, then the same construction being made

By Prop. VII $\mathrm{VH.HA} : \mathrm{EH.HG} = \mathrm{DN.NE} : \mathrm{YN.NY'}$
By parallels and compounding ratios $\mathrm{EH.HG} : \mathrm{DC.CK} = \mathrm{ET.TG} : \mathrm{DT.TK}$
By Prop. VII $= \mathrm{YN.NY'} : \mathrm{XN.NX'}$
Also by the same $\mathrm{DC.CK} : \mathrm{AC.CV} = \mathrm{XN.NX'} : \mathrm{DN.NE}$
Therefore ex æquali $\mathrm{VH.HA} : \mathrm{AC.CV} = \mathrm{DN.NE} : \mathrm{DN.NE}$
Therefore $\mathrm{VH.HA} = \mathrm{AC.CV}$.

But AH = AC. Therefore VH = CV which is absurd. Therefore CAH does not meet the section in any other point except A; that is, it is a tangent at A.

Art. 23.

FIGS. 20 & 21. Let now AA′ be the axis of the sections (Ellipse and Hyperbola) and O the middle point of AA′. Through O draw any line POp meeting the curves in P and p. Draw PN, pn perpendicular to AA′ and the tangents PT, pt meeting AA′ in T and t, also join PF, PF′, pF, pF′.

Then by Prop. II $\mathrm{AO}^2-\mathrm{ON}^2 : \mathrm{AO}^2-\mathrm{On}^2 = \mathrm{PN}^2 : \mathrm{pn}^2$
By sim. tri. $= \mathrm{ON}^2 : \mathrm{On}^2$
Therefore $\mathrm{AO}^2 : \mathrm{ON}^2 = \mathrm{AO}^2 : \mathrm{On}^2$
And therefore ON = On; hence (by Eu. I. 26) OP = Op.

Since therefore FF′, Pp bisect each other at O, the figure FPF′p is a parallelogram, and the angle OPF = OpF′. But both in the Ellipse and Hyperbola the angle FPT = F′pt. Therefore in the one case by addition and in the other by subtraction, the angle OPT is equal to the angle Opt. Therefore the tangents at the extremities of any line drawn through the centre of the axis to meet the sections in P and p are parallel to one another.

Art. 24.

Since therefore the points of contact of parallel tangents are in the same right line with the middle points of chords parallel to them, and also lie on right lines passing through the centre of the axis, we have the following :—

Figs. 18, 19, 20, 21, Def.—All lines such as Pp and AA, passing through the centre of the Axis of a Conic section, and bisecting chords drawn in the section parallel to the tangents at their extremities, are called Diameters of the section, and the chords they bisect are called Ordinates to the same, and as all diameters pass through one point, viz., the centre of the axis, this point is called the Centre of the section.

Note.—In the Parabola, since the axis only meets the Cone in one point, the other extremity being infinitely distant therefrom, for the section being parallel to a side of the Cone may be infinitely produced. The centre of the axis will therefore be infinitely distant from the vertex A, consequently all diameters of a Parabola passing through the same point at infinity are parallel to one another—and the reason why the tangent at any point on a Parabola bisects the angle contained by the focal distance and the diameter through the point of contact is apparent.

Art. 25.

Def.—In the Ellipse and Hyperbola if a diameter be parallel to the tangent at the extremity of another diameter, it is said to be conjugate to that other diameter.

Figs. 23 & 24. If a diameter be conjugate to another diameter, then also this other diameter shall be conjugate to the first. Let PCp be conjugate to the diameter DCd, and therefore parallel to the tangent at D; DCd, will also be parallel to the tangent at P.

Take any point Q in the section, and draw QUR parallel to PCp. Therefore QR is parallel to the tangent at D, and is therefore by this (Prop. VIII.) bisected in U. Join RC and produce it to meet the section in r, and join Qr, meeting Pp in V. Then since $QU = UR$ and $RC = Cr$, Qr is parallel to CD, and $QV = Vr$, therefore Pp bisects chords parallel to CD, and therefore CD is parallel to the tangent at P, that is, DCd is conjugate to PCp.

Note.—That all diameters of an Ellipse pass through one point may be proved otherwise, thus :— (Fig. 22.)

For if they do not pass through the same point let the diameter RR′ have its centre at L which is not the same point as O the centre of AA′. Let RR′ meet AA′ in Q, and draw WQW′ parallel to the tangent at A. Then by Prop. VIII. $WQ = W'Q$ and therefore by (Eu. I. 26) $QS = QS'$ and $WS = WS'$

But by Prop. VII. $AO^2 - OS^2 : AO^2 - OS'^2 = WS^2 : W'S'^2$
Therefore $AO^2 - OS^2 = AO^2 - OS'^2$ and $OS' = OS'$

Therefore QS is greater than QS′, but they been shown to be equal, which is absurd. Therefore RR′ and AA′ have not their centres at different but at the same points.

This method of proof is inapplicable to the Hyperbola, since the line WQW′ does not meet the curve.

Art. 26.

In the Ellipse and Hyperbola the axis is frequently termed the axis major, and the diameter at right angles to it the axis minor, and these are conjugate to one another.

Figs. 4 & 5. In the Ellipse it is easily shown that $AO^2—OF^2 = OB^2$, where OB is the semi axis minor. For $AO^2—OF^2$ = AFFA′ (by Eu. II. v.) = AH, A′H′ (by construction) but if through O a parallel to AD or AE be drawn, the segments of this parallel are respectively equal to AH and A′H′, and their rectangle is equal (by Prop. II.) to OB^2.

Now in the Hyperbola if a diameter meet the curve, its conjugate will not meet either branch, for it is parallel to the tangent at the extremity of the other diameter. This is true for any pair of conjugate diameters of an Hyperbola, and therefore for the axes; but although the axis minor of the Hyperbola does not meet the curve, yet its square CB^2 (Fig. 24) AF.FA′ as in the case of the Ellipse.

Figs. 23 & 24. If from the extremities of a diameter of an Ellipse or Hyperbola lines be drawn to any point in the section, the lines so drawn are called supplementary chords, and it is evident from Art. 25, that a pair of diameters parallel to any pair of supplementary chords will be conjugate to one another.

Hence in the Hyperbola (Fig. 24) if any diameter as CD meet both branches of the curve, all parallels to it as Qr drawn between the branches are bisected by its conjugate CP, to which the parallels are therefore ordinates.

Fig. 23.

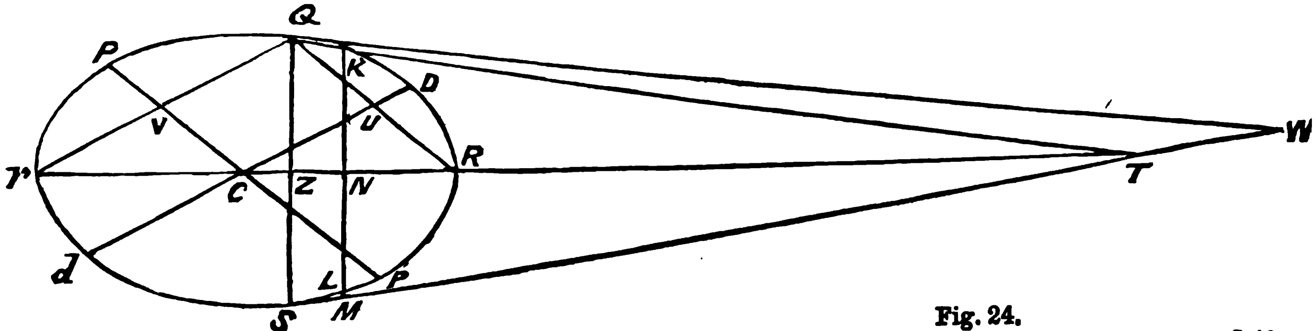
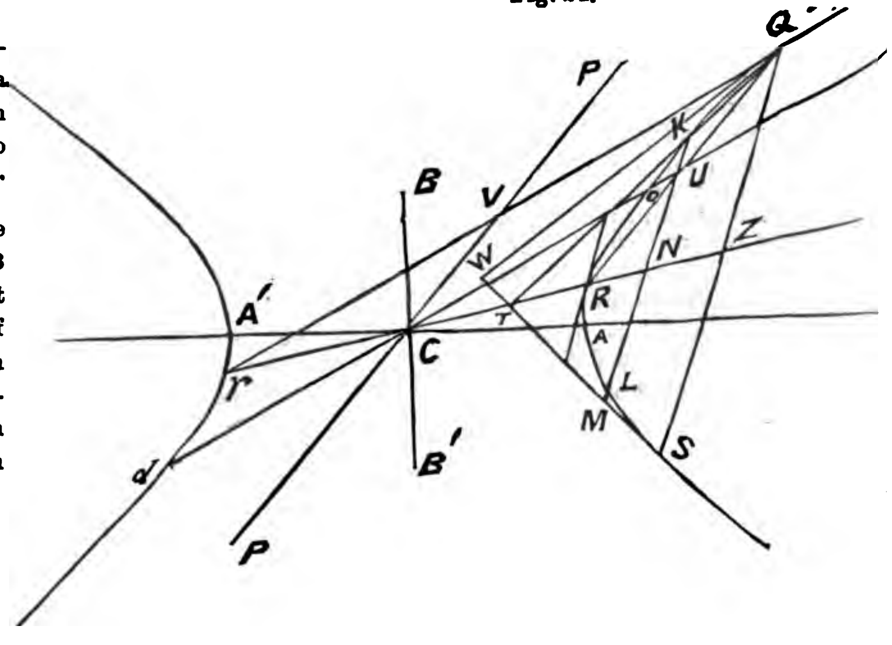


Fig. 24.



Cor.—Tangents at the extremities of any chord of a conic intersect each other on the diameter produced, to which the chord is an ordinate.

For let QS be any ordinate to the diameter CR (figs. 23 and 24) then the tangents at Q and S meet on CR. For if not let them meet at W, which is not on CR, and let the tangent SW meet CR in T. Join TQ meeting the curve again in K, and draw KNLM parallel to QS, meeting CR in N, the curve in L, and WS in M. Then since CR bisects QS, it also by Prop. VIII. bisects KL. Therefore $KN = NL$; but by the triangle TQS, since KM is parallel to QS, therefore $KN = NM$, and therefore $NL = NM$, which is absurd. Therefore the point W cannot fall otherwise than upon CR; that is, tangents at the extremities of the chord QS intersect on the diameter CR produced.*

Art. 27.

THE LATUS RECTUM.

Figs. 25 & 26. In the circle the square of the semi-ordinate is equal to the rectangle of the segments of the diameter to which it is at right angles. In the Conic sections this is not the case, but if in the Ellipse and Hyperbola AB, DE be any pair of conjugate diameters and AZ be a third proportional to AB, DE. Then if AZ be placed at right angles to AB, and BZ be joined, and from any point P in the section PN be drawn an ordinate to AB and therefore parallel to DE, and NL be drawn parallel to AZ, and meeting BZ in L. The rectangle AN.NL will equal NP^2.

For as $AB : DE = DE : AZ$

Therefore $AB^2 : DE^2 = AB : AZ$

But by Prop. VII. $AB^2 : DE^2 = AN.NB : NP^2$

And by sim. tri. $AB : AZ = BN : NL$

$= BN.NA : NA.NL$

Therefore ex æquali $AN.NB : NP^2 = AN.NB : NA.NL$

Therefore $NP^2 = AN.NL.$

Art. 28.

Figs. 25 & 26. The ancient Geometer Apollonius, called the first of the three proportionals AB, the transverse side of the figure, and the third proportional AZ, the right side or Latus Rectum of the figure.

When the diameters AB, DE coincide with the axis XOX′ and its conjugate YOY′ at right angles to it, then the Latus Rectum of the axis XOX′ is equal to the ordinate RFS drawn through the focus F

For by Prop. VII. $XO^2 : OY^2 = XO^2 - OF^2 : RF^2$

† Since $XO^2 - OF^2 = YO^2$ $\quad = YO^2 : RF^2$

Therefore $XO : OY = OY : RF$

Or $XX' : YY' = YY' : SR$

Therefore SR equals the Latus Rectum of the axis.

---

* The Figure for the Parabola is omitted.

† In the Hyperbola, $OF^2 - OX^2 = OY^2$. See Art. 26.

Art. 29.

Fig. 27. In the case of the Parabola it has been shown in Prop. V, that if P be any point on the curve and PN an ordinate to the axis that $PN^2 = 4AF.AN$. 4AF is therefore the Latus Rectum of the axis of the Parabola, and if BFC be an ordinate to the axis through the focus F, then BC may be shown to be equal to 4AF. For $BF^2 = 4AF^2$ therefore $BF = 2AF$, and $BC = 4AF$.

Art. 30.

Fig. 27. Again, let RR′ be any other diameter and DFE, PKp ordinates thereto, the former of which passes through the focus. Let the tangents at D and E (Prop. V.) meet on the directrix at H, and let the tangent at R meet the tangent at D in the point V, and the axis in T, and join FR. Then since (by Prop. V.) the angle $TRF = HRT = RTF$, therefore $FR = FT$, therefore also $HR = RN'$, that is $HN' = 2HR$, hence also $DN' = 2VR = 2HR = 2FR$

Therefore $DN'^2 = 4FR^2 = 4FR.RN'$ since $FR = RN'$

And by Prop. VII. Art. 21 $DN^2 : PK^2 = RN' : RK$

$= 4RF.RN' : 4RF.RK.$

But DN'^2 has been shown to be equal to $4RF.RN'$, therefore $PK^2 = 4RF.RK$.

Therefore 4RF, which is equal to DE, (for $DE = 2DN' = 4VR = 4HR = 4FR$) is the Latus Rectum of the diameter RR′. Hence the Latus Rectum of any diameter is equal to the focal ordinate of that diameter, or to four times the distance of its vertex from the focus.

Note.—The Latus Rectum of any diameter is sometimes called the Parameter. The term Latus Rectum being generally used in reference to the principal diameters or axes of the sections.

In the Parabola (Fig. 27) the square of the semi-ordinate (PK^2) is shown to be equal to the rectangle, under the absciss RK and the Parameter of the diameter RR′. In the the Ellipse and Hyperbola (Figs. 25 and 26) the square of the semi-ordinate (NP^2) is shown to be equal to the rectangle AN.NM, in the one case deficient from the rectangle AN.AZ, and in the other case exceeding the rectangle AN.AZ by the rectangle MQ, which is similar to AB.AZ. For these reasons, Apollonius named the sections Parabola, Ellipse, and Hyperbola, respectively.

Art. 31.
Fig. 28. The following are useful properties relating to the Latus Rectum of a Conic.

The semi Latus Rectum is a harmonic mean between the two segments of any focal chord of a Conic. Let PFp be a focal chord and draw the ordinates PN, pN′, then by similar triangles

$$\begin{aligned} FP : Fp &= FN : FN' \\ &= NX - FX : FX - N'X \\ &= FP - FB : FB - Fp. \end{aligned}$$

Therefore FP, FB, Fp are in harmonic progression and FB is a harmonic mean between FP and Fp.

Art. 32.
Fig. 28. Let a tangent at any point P of a Conic meet the Latus Rectum of the axis in D, and the directrix in T. Then $FD : FT = FA : AX$.

Draw PM perpendicular to the directrix, and join FM. Then since the angles TFP, TMP are right, a circle may be described about FTMP; therefore the angles FTP, FMP are equal, and the angles DFT, MPF are equal, being the complements of the angle DFP. Therefore the triangles DFT, MPF are similar.

Therefore
$$\begin{aligned} DF : FT &= FP : PM \\ &= FA : AX. \end{aligned}$$

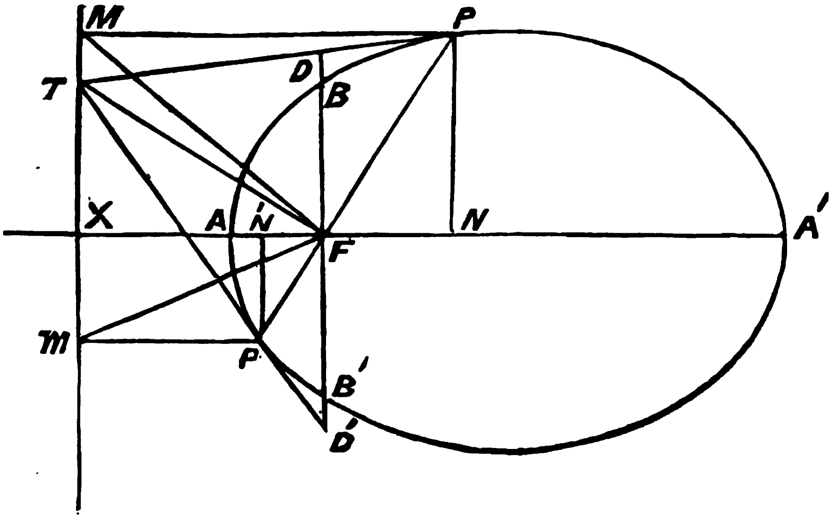
Fig. 28. Again, if PF be produced to meet the curve in p, and Tp be joined, Tp will be a tangent to the curve. Let Tp be produced to meet the Latus Rectum produced in D′. Then it may be shown in the same manner that

$$\begin{aligned} FD' : FT &= Fp : pm \\ &= FA : AX \end{aligned}$$

Wherefore $FD : FT = FD' : FT$

Therefore $FD = FD'$

Fig. 28.



THE GENERATING OR AUXILIARY CIRCLE OF A CONIC.

Art. 33.
Figs. 29, 30, & 31.

If AA′ be a Conic section of which XX′ is the directrix, and F the focus, and if round any point S in the plane of the Conic a circle be described, the radius of which has the same ratio to the distance of S from the directrix that FA has to AX. (That is if SR : SZ = FA : AX.) The circle thus described is called the Generating or Auxiliary circle of the Conic belonging to the point S.

Take any point Q on the directrix, and draw any line Qmn cutting the circle in the points m and n. Join SQ, and draw FP parallel to Qm, and meeting the directrix in P, also draw PMN parallel to QS. Join Sm, Sn, and draw FM, FN respectively parallel to them. Then M and N are points on the Conic. For draw MD, ND′, md, nd′ perpendicular to the directrix.

Then by similar triangles $\quad FM : MP = Sm : SQ$
And $\quad MP : MD = SQ : SZ$
∴ Ex æquali $\quad FM : MD = Sm : SZ = FA : AX$

Therefore (by Prop. V.) the point M is on the curve, and in like manner the point N may be shown to be thereon.

The radius of the generating circle of any given point S may easily be determined thus. Join AS and produce it to meet the directrix, and join the point in which it meets the directrix with the focus F, and produce this line to meet the perpendicular SZ in K, then SK is the radius required.

Art. 34.
Figs. 29, 30, & 31.

Since Q is any point on the directrix, suppose Q to have such a position thereon as V or V′; let the points m and n coincide at R and R′ so that VR, V′R′ are tangents to the circle, and also let V and V′ be so taken that VR, V′R′ both pass through F. Then if VS, V′S be joined and produced, these lines will be tangents to the Conic.

For draw FY, FY′ parallel to SR, SR′ and meeting SV, SV′ in Y and Y′; then the points Y and Y′ are on the curve. Draw the perpendiculars YK, Y′K′ on the directrix,

Then by similar triangles $\quad FY : SR = YV : SV = YK : SZ$
Therefore $\quad FY : YK = SR : SZ = FA : AX.$

Therefore Y is on the curve, as in like manner Y′ may also be shown to be; but every other point of the lines VY, V′Y′, except Y, Y′, is without the curve. *

---

* The Student will observe that if any two points be taken on a Conic, and any two parallels be drawn from them to the directrix, the distances of the points from the focus are to one another as the parallels, for by similar triangles the latter are as the perpendiculars from the same points on the directrix, as examples see (Figs. 29, 30, and 31) where

$$FY : YW = FY' : Y'W.$$

For (Fig. 29) if possible let VY meet the Conic again in G. Join FG, and draw GH parallel to FY, and therefore perpendicular to FV, also draw GL at right angles to the directrix. Then since the points G and Y are on the Conic.

$FG : GL = FY : YK$[†]

Alternately and by sim. tri. $FG : FY = GL : YK = GV : YV = GH : FY$

That is $FG : FY = GH : FY.$

Therefore $FG = GH$, which is impossible, since GH is at right angles to FV and therefore FG is greater than GH. Therefore VY does not meet the curve again in any other point G; therfore it touches in Y. That is, VY is a tangent to the Conic, similarly also Y'V' is a tangent.

Hence to draw tangents to a Conic from a given point S without it. Describe a generating circle round S, and draw the tangents FR, FR' thereto and let them produced meet the directrix in V and V'. Join VS, V'S, these lines when produced will be the tangents required.

There are other methods of drawing tangents to each of the sections which will be given afterwards.

Cor. 1.—From the construction of the Art., it is evident that the angles YFV, Y'FV' are right angles; and if YV, Y'V' be produced to meet the other directrix in V_2, V_3 and $F'V_2$, $F'V_3$ be joined, the angles $YF'V_2$, $Y'F'V_3$, are also right (as may be shown in a similar manner by drawing round S a generating circle, the radius of which has the same ratio to its distance from the directrix belonging to the focus F' that AF has to AX. And hence the Student will observe that every point—except the centre of the axis which has only one—has two generating circles pertaining to it, each of which is associated respectively with one of the foci and its allied directrix.)

Cor. 2.—Let the tangent at Y, meet the directrix belonging to F' in V_2, and draw HYh perpendicular to the directrices and meeting them in H and h. Then

$FY : YH = FA : AX = F'Y : Yh$

Therefore $FY : F'Y = YH : Yh$

By similar triangles $= YV : YV_2$

But the angles YFV, $YF'V_2$, are equal being right, therefore (by Eu. VI 7) the triangles VFY, $V_2F'Y$ are similar, and the angle $FYV = F'YV_2$; that is, the focal distances make equal angles with the tangent.

Note.—The principle of the generating circle is founded upon the property contained in Cor. 1 to Prop. V, and in the two corollaries just given to (Art. 34) it is shown how to derive by the assistance of this principle two of the properties that are demonstrated in Prop. V. In the next Art., the second part of Prop. V. will be proved by aid of the same principle.

[†] The lines YK, Y'K', GL are omitted in the Figures.

Art. 35.

If a right line be drawn in an Ellipse or between opposite branches of an Hyperbola, the sum or difference of the lines joining the extremities of the chord thus drawn to one of the foci will be equal to the diameter of the generating circle belonging to the centre of the chord. See Figs. 35 and 36.

Let MN be a chord of an Ellipse or Hyperbola, and let it be produced to meet the directrix in P. Join PF, cutting the generating circle drawn round S, the middle point of MN in m and n. Join Sm, Sn, FM, FN, and draw MK parallel to PF and meeting FN in K; also join Mn, and produce it to meet NF produced in L.

Then by the property of the generating circle, Sm is parallel to FM, and Sn to FN; and it is evident that the angles MFP, NFm are equal, for they are equal to the equal angles Snm, Smn, and therefore, since MK is parallel to PF, the angles FMK, FKM are equal, for they are alternate to MFP and NFm; therefore FK = FM; and since NS = SM, and Sn is parallel to NL, therefore Mn = nL; and therefore also FL = FK = FM. Hence NL is equal to the sum or difference of FN and FM. But NL is equal to twice Sn, that is to the diameter of the generating circle about S; therefore the sum or difference of FN and FM is equal to the diameter of the same circle.

Figs. 37 & 38 Let now the line MN pass through the centre of the axis of the Ellipse or Hyperbola, and let S be the centre, and produce the line MN to meet both the directrices in P and P′ respectively, and complete the figures. Then plainly SP = SP′ and SF = SF′, and the angles FSP, F′SP′ are equal; therefore (by Eu. I. 4) the triangles FSP, F′SP′ are equal, and the angle SFP = SF′P′; and since Sn = Sm′ (for they are the radii of the circle) and FS = F′S, therefore Sn : SF = Sm′ : SF′, and the angles SnF, Sm′F′ are either both less or both greater than a right angle, therefore (by Eu. VI. 7) the triangles FSn, F′Sm′ are similar, and the angle FSn, is equal to the angle F′Sm′; therefore nSm′ is a straight line, and in the same manner mSn′ may be shown to be a straight line. But by the property of the generating circle, NF is parallel to Sn and F′M to Sm′ therefore FN is parallel to F′M; similarly it may be shown that FM is parallel to FN′, therefore the figure FNF′M is a parallelogram; and by what has been proved in the first part of this Art., the sum or difference of FN and FM, and also of F′M and F′N, is equal to the diameter of the generating circle about S, that is to AA′ the axis of the section. Therefore the sum or difference of the distances of the foci from any point on an Ellipse or Hyperbola is equal to the axes of those curves.

Cor.—Since FMF′N has been shown to be a parallelogram and FF′, NSM are its diagonals, it is therefore evident from this Art. that all lines drawn through the centre of the axis of an Ellipse or Hyperbola are bisected at the centre. Compare Art. 23.

Art. 36.

The principle of the generating circle, and its application to drawing tangents to Conics as just explained, and also to expound the general theory of the Conic sections was first given by the Rev. G. Walker, F.R.S., in his work on Conics, part of which was published in 1794. The generating circle was so named by its discoverer for the following reason.

Figs. 32, 33, & 34. If XX′ be a straight line, and S, F any two points without it, then if a circle be described round S, and a point P move along XX′, whilst PS, PF revolve about S and F. If m, n be the intersections of PF with the circle, and Sm, Sn be joined, and FM, FN be drawn parallel to them meeting SP in M and N, the locus of the points M and N is a Conic section of which XX′ is the directrix and F the focus; and it will be an Ellipse, Parabola, or Hyperbola, according as the radius of the generating circle about S is less than, equal to, or greater than the distance of the point S from the line XX′. This is evident for if SZ, MD be drawn perpendicular to XX′ then by similar triangles

$$\text{Sm} : \text{FM} = \text{SP} : \text{MP} = \text{SZ} : \text{MD}$$

Therefore alternately

$$\text{Sm} : \text{SZ} = \text{FM} : \text{MD}$$

Therefore FM has to MD the constant ratio of Sm to SZ, which is a general property of the Conic sections deduced in Prop. V. Hence the locus of M and also of N is a Conic section, the focus of which is F, and XX′ the directrix; and it will be an Ellipse, Parabola, or Hyperbola according as the radius of the generating circle is less than, equal to, or greater than SZ the distance of its centre from XX′. (See Cor. 1. Art. 15.)

Art. 37.

If the perpendicular from S, the centre of the generating circle on XX′, passes through F, and its radius is a mean proportional between the distances of its centre from the focus and directrix, then the diameter of the circle is in the cases of the Ellipse and Hyperbola equal to the axis of the section, and the circle becomes what in those cases has been termed by writers on the Conic sections the Auxiliary circle of the Conic. (See. Cor. Art. 15)

Art. 38.

The radius of the generating circle of the middle point of any chord of a Conic passing through the focus is equal to half the said chord.

Figs. 32, 33, & 34. Let NL be a chord passing through the focus (Figs. 32, 33, & 34,) and O its middle point. Draw Ll, Om, Nn,* perpendicular to the directrix. Then (by Art. 15, Cor. 1)

$$\text{FL} : \text{Ll} = \text{FN} : \text{Nn}$$

Alt$^{\text{do}}$ comp$^{\text{do}}$ et alt$^{\text{do}}$

$$\begin{aligned} \text{FL} : \text{Ll} &= \text{NL} : \text{Ll} + \text{Nn} \\ &= \tfrac{1}{2}\text{NL} : \tfrac{1}{2}(\text{Ll} + \text{Nn}) \\ &= \tfrac{1}{2}\text{NL} : \text{Om, since Om} = \tfrac{1}{2}(\text{Ll} + \text{Nn}). \end{aligned}$$

Hence, $\frac{1}{2}$NL is equal to the radius of the generating circle about the middle point O; the radius therefore of the generating circle of the Focus is equal to the semi Latus Rectum of the axis, and in the Parabola this circle touches the directrix.

It will also be observed that the two generating circles of any point on a Conic pass through the foci, and the generating circle of any point on a Parabola touches the directrix.

Art. 39.

Fig. 39. The radius of the generating circle of the Vertex of the axis is equal to the distance of the vertex from the focus,† and the construction of a Conic by means of the generating circle of this point is identical with that employed by Mr. Besant in his work on the Conic sections, (see Fig. 39) where (as in Art. 36) FM is parallel to Am or Sm, and M is consequently a point on the curve; and if A′P be joined, and MF be produced to meet A′P in N, N is also a point on the curve; for

$$\text{FM} : \text{Am} = \text{PM} : \text{AP} = \text{KM} : \text{AX}$$

Therefore

$$\text{FM} : \text{KM} = \text{AF}(= \text{Am}) : \text{AX}$$

---

* Ll, Om, Nn, are omitted in the Figures.

† Or more generally, the radii of the generating circles of the vertices of the axis are equal to the distances of the vertices from the foci.

Also, since AF = Am, and the angle AFm = AmF = mFN, therefore PF bisects the angle XFN or MFA′; and if NL′K′ be drawn perpendicular to the directrix, then L′N will equal FN, and by parallels

$$NF(= NL') : NK' = FA' : A'X$$

Therefore (by Cor. I. Art. 15) the point N is also on the curve.

Art. 40.
Figs. 32, 33, & 34. It is also evident from the demonstration of the above locus that the angles MFn, NFm are equal, for they are equal to the equal angles Smn, Snm; and therefore, if NF be produced to meet the section in L, PF bisects the angles MFL, or the latter conclusion may be shown thus; draw MD, ND′, perpendiculars on the directrix, then

$$FM : FN = MD : ND'$$

By similar triangles $= MP : NP$

Therefore PF bisects the angle MFL (by Eu. VI. A.)

Art. 41.
Figs. 29, 30, & 31. Further it is also evident that the tangents from S subtend equal angles at the focus F. For FY is parallel to SR and FY′ to SR′; therefore the angle SFY = FSR and SFY′ = FSR′; but FSR = FSR′ by the circle; therefore SFY = SFY′.

Observe when the tangents touch the same branches of the Hyperbola the angles SFY, SFY′ are equal, but when the tangents touch opposite branches, the angles SFY, SFY′ are supplementary; for (Fig. 31) the angle SFY′ = FSR′, and SFY is the supplement of FSR, that is of FSR′, and therefore of SFY′

Art. 42.
Figs. 32, 33, & 34. Join PL and produce it to meet MF produced in K. Then since PF bisects the angle MFL, we have

$$KF : FL = KP : PL$$

Alternately $KF : KP = FL : PL$

But the point L is on the curve, therefore the point K is also. From this it is manifest that the lines joining the extremities of focal chords meet on the directrix. *

Art. 43.
Figs. 29, 30, & 31. Since SF bisects the angle RFR′, we have the following Theorem. If tangents be drawn from any point S to a Conic, and be produced to meet the directrix in V and V′, then if SF, VF, V′F be joined, SF will bisect the angle contained by VF and V′F or the supplement of that angle.

Art. 44.
Figs. 32, 33 & 34. If a Conic and its foci and directrix be given, the points in which it is cut by a given straight line can be found. Let the Conic AA′ be cut by a given straight line PMN. Then if round any point S in PMN a generating circle be drawn cutting FP in m and n, and Sm, Sn be joined, and FM, FN be drawn parallel to them, these lines will intersect the Conic and also the line PMN in the required points M and N.

---

* See note at foot of page 26.

Art. 45.
FIGS. 29, 30, & 31.

If YY′ be joined and produced to meet the directrix in W, and WF be joined, then WF will be at right angles to SF. * Draw Yk, y′k′ perpendicular to the directrix, then Y′F : YF = Y′k′ : Yk = Y′W : YW by similar triangles. Therefore WF bisects the supplement of the angle YFY′, and it has been shown that SF bisects YFY′. Therefore SF is at right angles to FW. Observe here, also, that when the points Y, Y′ lie on opposite branches of the Hyperbola, WF bisects the angle YFY′.

It is shown in Cor. 1, Art. 34, that if a tangent be drawn to a Conic and produced to meet the directrix, that the lines drawn from the focus to the point in which it meets the directrix and to the point of contact, are at right angles to one another; since, therefore SF is at right angles to FW, the tangents from W meet the Conic in the points in which SF meets it. Let SF meet YY′ in T, then since TF bisects the angle YFY′ and WF bisects the supplement of that angle

$$\mathrm{YT} : \mathrm{TY'} = \mathrm{YF} : \mathrm{Y'F} = \mathrm{YW} : \mathrm{Y'W}$$

Therefore YY′ is harmonically divided in T and W. Hence, if from any point in the directrix a pair of tangents be drawn to a Conic and the chord of contact joined, any line drawn cutting the Conic and so as to pass through the point on the directrix will be harmonically divided by this point and its intersection with the chord of contact of tangents therefrom. This will be shown afterwards to be true when the point from which the tangents are drawn is not situated on the directrix.

Art. 46.

NOTE.—The general equation for any Conic section may be obtained by aid of the generating circle as follows :—

Let the equation of XX′ be $x \cos\alpha + y \sin\alpha - \delta = 0$.

Let the co-ordinates of S *be* $(a\ b)$, *of* F *be* (p, q), *and of* M *be* (xy) ;

Then by the nature of the generating circle—

$$R : a\cos\alpha + b\sin\alpha - \delta = \sqrt{(x-p)^2 + (y-q)^2} : x\cos\alpha + y\sin\alpha - \delta$$

Therefore $R^2\ (x\cos\alpha + y\sin\alpha - \delta)^2 = \left\{(x-p)^2 + (y-q)^2\right\}\ (a\cos\alpha + b\sin\alpha - \delta)^2$

Reducing this becomes—

$$\left\{R^2\cos^2\alpha - (a\cos\alpha + b\sin\alpha - \delta)^2\right\} x^2 + \left\{R^2\sin^2\alpha - (a\cos\alpha + b\sin\alpha - \delta)^2\right\} y^2$$
$$+ 2R^2\sin\alpha\cos\alpha\, xy + \left\{2p\,(a\cos\alpha + b\sin\alpha - \delta)^2 - 2R^2\,\delta\cos\alpha\right\} x$$
$$+ \left\{2q\,(a\cos\alpha + b\sin\alpha - \delta)^2 - 2R^2\,\delta\sin\alpha\right\} y + R^2\delta^2 - (p^2 + q^2)\,(a\cos\alpha$$
$$+ b\sin\alpha - \delta)^2 = 0.$$

Writing this in an abbreviated form it becomes—

$$(R^2\cos^2\alpha - P^2)\,x^2 + (R^2\sin^2\alpha - P^2)\,y^2 + 2R^2\sin\alpha\cos\alpha\, xy + Dx + Ey + F = 0$$

And applying the test $B^2 - 4AC$ *(see Salmon's Conics, Chap. X.)*

$$B^2 - 4AC \equiv 4R^4\sin^2\alpha\cos^2\alpha - 4(R^2\cos^2\alpha - P^2)\,(R^2\sin\alpha - P^2$$
$$\equiv 4(R^2 - P^2)\,P^2$$

And this will be negative, zero, or positive, according as R *is* $< \; = \;$ *or* $>$ P; *that is, since* P *represents the perpendicular from the centre of the generating circle on the directrix, according as the section is an Ellipse, Parabola, or Hyperbola.*

---

* The lines Yk, Y′k′ are omitted in the figures.

Art. 47.

PROP. IX. THEOR.—*If a Trapezium be inscribed in a Conic section and any point E be taken in the section, and through E two lines are drawn parallel to any two adjoining sides of the Trapezium, and intersecting the opposite sides in the points N, Q, and H, R; then the rectangle ER.EH will have to the rectangle EN.EQ a constant and given ratio.*

Also if from any two points E and e on the curve four pairs of parallels, viz., EL, el; EK, ek; EO, eo; EM, em, be drawn in any angles whatever to the sides of the Trapezium meeting them in the points L, l; K, k; O, o; Mm,. Then EL.EM : el.em = EK.EO : ek.eo.

FIGS. 40, 41, & 42. Let ABDC be the inscribed trapezium; and let EN, EH be parallel to the adjoining sides AB, AC. Draw DFG, BPV parallel to AC. Join CV meeting EH, DG in T and S. Draw the diameter XY to which AC in an ordinate, and which will therefore bisect the parallels DF, BV, EZ, SG, TR. Hence, therefore SD = FG and TZ = ER; as would be the case likewise if SG, RH were tangents. Therefore

By similar triangles $HT : SD = CT : CS$
And parallels $= AR : AG$
Alternately $HT : EQ(= AR) = FG(= SD) : AG$
Again $ER(= BP) : PN = DG : GB$
Therefore $HT.ER : EQ.PN = DG.GF : AG.GB$
But by Prop. VII. $DG.GF : AG.GB = ER.RZ : AR\ RB$
That is $= ER.ET : EQ.EP$
∴ Comp^{do} $DG.GF : AG.GB = ER.ET + ER.HT : EQ.EP + EQ.PN$
$= ER.EH : EQ.EN$

But DG.GF : AG.GB is a given and constant ratio, since the points A, B, C, D, are fixed. Therefore ER.EH has to EQ.EN a given and constant ratio.

Next to prove the latter part of the Theorem—

Through E and e draw EQN, eqn, parallel to AB, and ERH, erh, parallel to AC.

Then by similar triangles $ER : er = EL : el$
And $EH : eh = EM : em$
∴ Compounding $ER.EH : er.eh = EL.EM : el.em$
Again by similar triangles $EQ : eq = EK : ek$
And $EN : en = Eo : eo$
∴ Compounding $EQ.EN : eq.en = EK.EO : ek.eo$

But by the first part of the Theorem already proved—

$ER.EH : er.eh = EQ.EN : eq.en$
Therefore $EL.EM : el.em = ER.EH : er.eh$
$= EK.EO : ek.eo$
Or alternately $EL.EM : EK.EO = el.em : ek.eo$

Art. 48.

Figs. 40, 41, & 42. Cor. 1.—From what was shown in the proof of the first part of the Theorem—

$$EH.ER : HT.ER = EQ.EN : EQ.PN$$

$\therefore$ $$EH : HT = EN : PN$$

Div^{do} $$EH : ET = EN : EP$$

And conversely, if EH be made to ET as EN to EP, and BP, CT, be joined, they will meet on the curve.

Art. 49.

Fig. 43. Cor. 2.—If two of the opposite sides AB, CD, be produced to meet in W, and the diagonals CB, AD, meet in Y; then if E and e be so taken that the line joining Ee passes through W and Y as in (Fig. 43); then if EL, el, EK, ek, EO, eo, EM, em, all lie in the line Ee we have—

$$EW : eW = EY : eY$$

For by this Theorem $EW^2 : eW^2 = EY^2 : eY^2$

Therefore $EW : eW = EY : eY$

Therefore Ee is harmonically divided in Y and W.

Art. 50.

Fig. 43. Cor. 3.—If AB, DC revolve about W so as to become tangents, and the diagonals AD, BC coincide and become the chord of contact, it will still be—

$$EW : eW = EY : eY$$

We have therefore the following theorem.

If tangents be drawn to a Conic from a given point, and the points of contact be joined, any line passing through the given point and cutting the curve will be harmonically divided; the four points of division being the given point and the intersections of the line with the curve and chord of contact. (See Art. 44.)

This may be proved otherwise by drawing parallels (to meet the tangents from W) through the points E and e, for the rectangles of the segments of these parallels are (by Prop. VII.) as the squares of the segments of the tangents between the parallels and the points of contact, and by similar triangles the rectangles of the segments of the parallels are to one another as the squares of the segments of the tangents between the parallels and the point W, hence the truth of the Cor.

Art. 51.

Fig. 43. Cor. 4.—It follows therefore from the last two Corollaries, that if WT, WT′ be two tangents to a Conic from a point W, and through W any two lines WBA, WCD be drawn to intersect the curve in the points A, B, C, D, that if AD, BC be joined they intersect each other on the line TT′.

Art. 52.

PROP. X. THEOR.—*If RP, RQ be two chords of a Conic of which F is a focus and XX′ the corresponding directrix. Then if RP, RQ be produced to meet the directrix in V and W. and QF, PF, VF, WF, be joined, the angle VFW equals half the angle QFP; also if tangents be drawn at P, Q, R, intersecting in K, L, M, the angle VFW is equal to the angle LFM.*

FIG. 44. For by Art. 40, VF, WF bisect the supplements of the angles RFP, RFQ. Therefore $\mathrm{WFR} = 90 - \tfrac{1}{2}\,\mathrm{RFQ}$, and $\mathrm{VFR} = 90 - \tfrac{1}{2}\mathrm{RFP}$, † hence subtracting

$$\mathrm{WFV} = \tfrac{1}{2}(\mathrm{RFP} - \mathrm{RFQ}) = \tfrac{1}{2}\mathrm{QFP}.$$

Again, the angle $\mathrm{LFM} = \mathrm{LFR} - \mathrm{MFR} = \tfrac{1}{2}(\mathrm{RFP} - \mathrm{RFQ})^{*} = \tfrac{1}{2}\mathrm{QFP}$

Therefore $\mathrm{LFM} = \mathrm{WFV}$.

Art. 53.

FIG. 45. NOTE.—In the Hyperbola it should be observed in reference to Art. 52, that when RP, RQ, are both drawn in the same branch, the angles WFV, ½QFP, are supplementary, but when the points P and Q′ lie on opposite branches, the same angles are complementary to one another, but whether P and Q are on the same or opposite branches, the angles LFM, WFV, are supplementary.

Let P and Q be on the same branch of the Hyperbola, then

$$\begin{aligned}\mathrm{WFV} &= \mathrm{WFR} - \mathrm{VFR} = (90 + \tfrac{1}{2}\mathrm{QFR}) - (90 - \tfrac{1}{2}\mathrm{RFP})\,\dagger\\ &= \tfrac{1}{2}(\mathrm{QFR} + \mathrm{RFP}) = 180 - \tfrac{1}{2}\mathrm{QFP}\end{aligned}$$

$\therefore\ \mathrm{WFV} + \tfrac{1}{2}\mathrm{QFP} = 180$. Therefore WFP, ½QFP, are supplementary.

Also $\mathrm{LFM} = \mathrm{LFR} - \mathrm{MFR} = \tfrac{1}{2}(\mathrm{RFP} - \mathrm{QFR})^{*} = \tfrac{1}{2}\mathrm{QFP}$

And therefore WFV, LFM, are supplementary

Again, let P and Q′ be now situated on opposite branches of the Hyperbola, then

$$\begin{aligned}\mathrm{W'FV} &= \mathrm{W'FR} - \mathrm{VFR} = \tfrac{1}{2}\mathrm{Q'FR}\,\ddagger - (90 - \tfrac{1}{2}\,\mathrm{RFP})\,\dagger = \tfrac{1}{2}(\mathrm{Q'FR} + \mathrm{RFP}) - 90\\ &= 180 - \tfrac{1}{2}\mathrm{Q'FP} - 90 = 90 - \tfrac{1}{2}\mathrm{Q'FP}\end{aligned}$$

$\therefore\ \mathrm{W'FV} + \tfrac{1}{2}\mathrm{Q'FP} = 90$, and therefore W′FV, ½Q′FP, are complementary.

$$\begin{aligned}\text{Also}\quad \mathrm{LFM'} &= \mathrm{LFR} + \mathrm{RFM'} = \mathrm{LFP} + 180 - \mathrm{Q'FM'} \quad \text{(by Art. 41.)}\\ &= \mathrm{Q'FP} + 180 - \mathrm{LFM'}\\ \text{Therefore}\quad 2\mathrm{LFM'} &= \mathrm{Q'FP} + 180\\ \text{And}\quad \mathrm{LFM'} - \tfrac{1}{2}\mathrm{Q'FP} &= 90.\end{aligned}$$

That is, the difference of the angles LFM′, ½Q′FP, is a right angle.

Therefore $\mathrm{W'FV} + \mathrm{LFM'} = 180$.

FIGS. 44 & 45. COR.—If the point R revolve round the Conic whilst P and Q remain fixed, either the angles LFM, VFW, remain constant, being always equal to ½QFP, or they are together equal to two right angles.

---

* By Art. 41. † By Art. 40. ‡ By Art. 45.

Fig. 44.

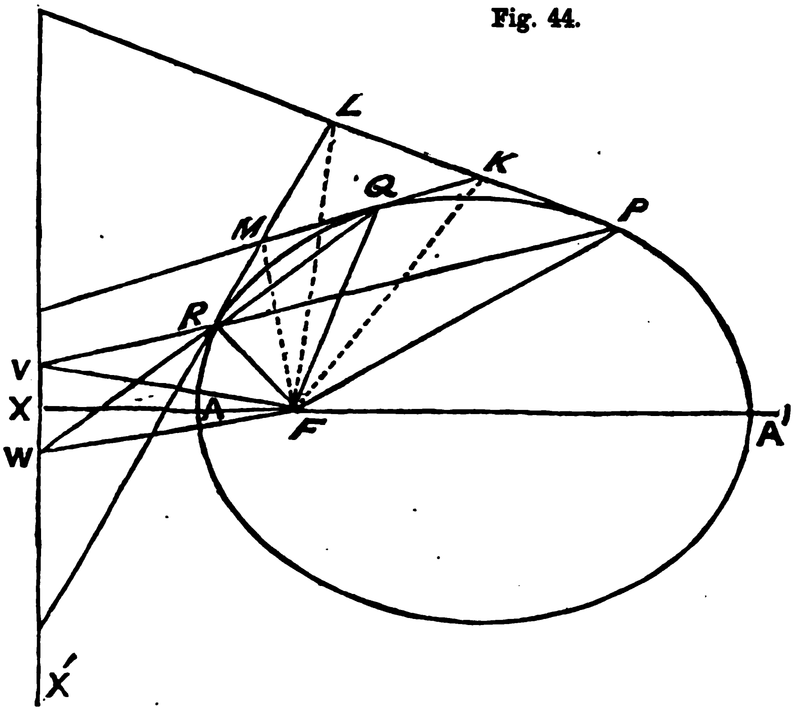
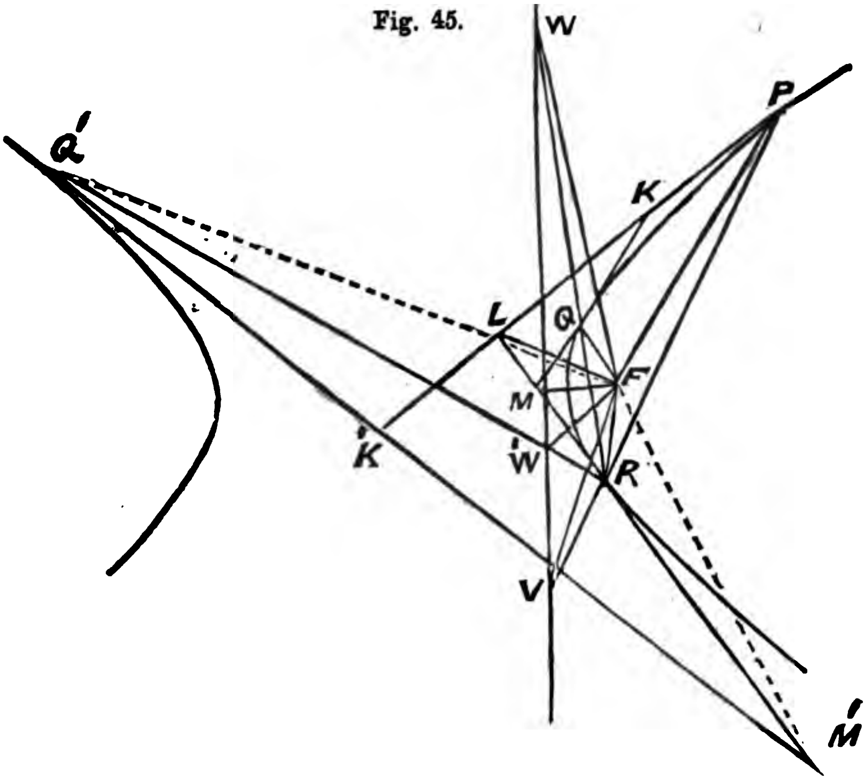


Fig. 45.



THE NORMAL TO A CONIC.

Art. 54.

DEF.—The Normal at any point of a Conic section is the straight line drawn through that point at right angles to the tangent at the point.

FIGS. 46 & 47. Let PG be the Normal and PT the tangent at the point P on the Conic AA′. Then if the Normal meet the axis in G

$$FG : FP = FA : AX.$$

Let the tangent at P meet the directrix in T. Join FT and draw the ordinate FD, meeting the tangent in D.

Then since FT is at right angles to FP (by Prop. V.) and PG to PT, the angle FPG will be equal to the angle PTF, and the angles FDT, PGF, are evidently equal, being the complements of the angle PKF; therefore the triangles FPG, DTF, are similar.

Therefore $FG : FP = FD : FT = FA : AX$ (by Art. 32.)

Art. 55.

FIGS. 46 & 47. If PF be produced to meet the section in p, and the Normal pG′ be drawn at p, and GL, G′L′, be drawn at right angles to Pp, then in the same manner it may be shown that

$$FG' : Fp = FA : AX$$

Therefore $FG : FP = FG' : Fp$

Inv^{do} and alt^{do} $FP : Fp = FG : FG'$

$= LG : L'G'$ by similar triangles.

$= FL : FL'$,, ,,

$= FP - PL : PL' - pF.$

But since the triangles TFP, PLG, are similar

$$TF : PL = FP : LG$$

And similarly $TF : pL' = Fp : L'G'$

Therefore $TF : PL = TF : pL'$

And therefore $PL = pL'$

Hence $FP : Fp = FP - PL : PL - pF$

Therefore FP, PL, Fp, are in harmonical progression.

Hence by Art. 31, PL is equal to the semi Latus Rectum.

Art. 56.

FIGS. 46 & 47. Again, since $FP : Fp = FP - PL : PL - Fp$

Therefore $FP\,(PL - Fp) = Fp\,(FP - PL$

Or $(FP + Fp)\,PL = 2FP.Fp$

That is $Pp.PL = 2FP.Fp.$

Or the rectangle of a focal chord and the semi Latus Rectum of the axis, is equal to twice the rectangle of the segments of the chord into which it is divided at the focus.

Fig. 46

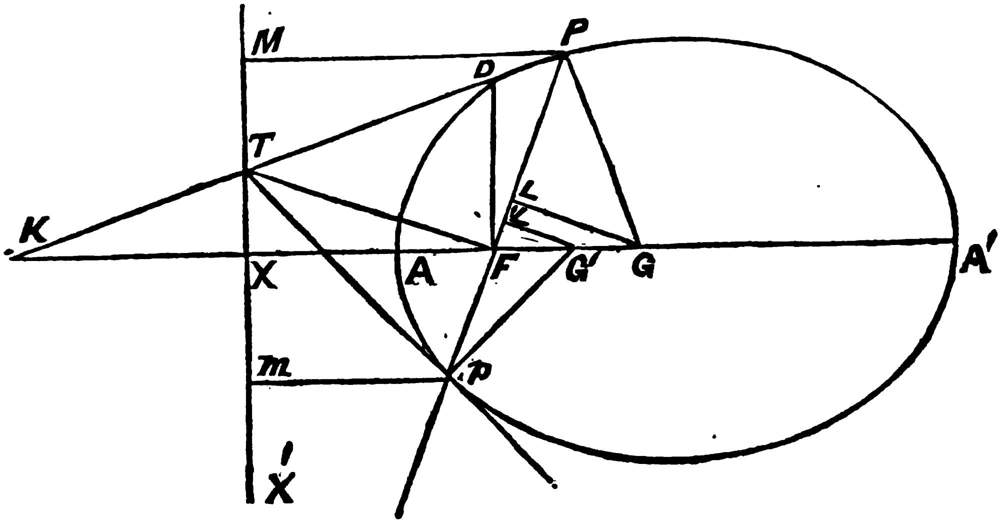
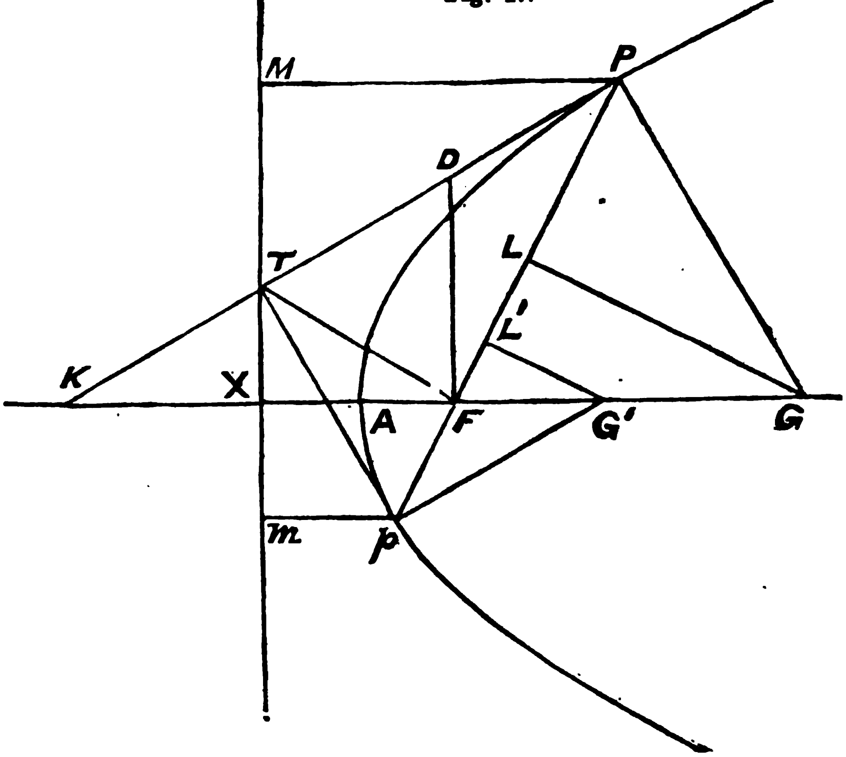


Fig. 47.



Art. 57.

Figs. 46 & 47. From the last Article it follows that the rectangles under the segments of any two focal chords are to one another as the chords themselves.

For let PFp, LFl, be two focal chords, then putting Z for the semi Latus Rectum

$$PF.Fp : LF.Fl = 2Pp.Z : 2Ll.Z = Pp : Ll.$$

Art. 58.

PROP. XI. THEOR.—*If a circle intersect a Conic section in four points, the several pairs of chords of intersection are equally inclined to the axis.*

FIGS. 48, 49, Let a Conic section of which the axis is AA', be cut by a circle in four points P, Q, R, S; the three pairs of chords joining the points P, Q, R, S, shall be be equally inclined to AA'

First, let the section be either an Ellipse or an Hyperbola, then if through the centre C, diameters CY, CX, be drawn parallel to any of the pairs of chords of intersection, to PR, SQ, for instance, which interest in O, then by Prop. VII

$$PO.OR : SO.OQ = CY^2 : CX^2$$

But $PO.OR = SO.OQ$ by the circle, therefore $CY^2 = CX^2$, and $CY = CX$, but equal diameters are equally inclined to the axis,* and therefore the chords PR, SQ, parallel to CY, CX, are also equally inclined to the Axis AA'; and similarly the other pairs of chords intersecting at O' and O'' may be shown to be equally inclined to AA'.

FIG. 50. In the Parabola, bisect PR, QS in E and H, and draw EZ, HW, OV, parallel to the axis AA', also draw VK, Vk† parallel to PR.QS and meeting EZ.WH in K and k, and join FZ, FW. Then by Art. 30, $PE^2 = 4FZ.ZE$, and $VK^2 = 4FZ.ZK$; therefore $PE^2 - EO^2 = 4FZ.VO$; similarly $SH^2 - HO^2 = 4FW.VO$, and therefore

$$\begin{aligned} PE^2 - EO^2 : SH^2 - HO^2 &= 4FZ.VO : 4FW.VO \\ &= FZ : FW. \end{aligned}$$

But $PE^2 - EO^2 = SH^2 - HO^2$ by the circle, that is, $PO.OR = QO.OS$. Hence, $FZ = FW$, and consequently FZ, FW, are eqnally inclined to the axis, and the points Z, W, are equidistant from the axis, and therefore the tangents at Z and W are equally inclined to the axis AA', and therefore also are the chords PR, QS, which are parallel to the tangents at Z and W.

NOTE.—The converse of this proposition is also true, that is, if two chords PR, QS, make equal angles with the axis of a Conic, a circle may be drawn through the points of intersection P, Q, R, S.

---

* That equal diameters of the Ellipse and Hyperbola are equally inclined to the axis may thus be shown. Let CX, CY be equal semi-diameters. Join FX, FY, $F'X$, $F'Y$, then since the points X and Y are on the curve, $FX + F'X = FY + F'Y$; therefore $FX^2 + F'X^2 + 2FX.F'X = FY^2 + F'Y^2 + 2FY.F'Y$. But $FX^2 + F'X^2 = 2CX^2 + 2CF^2 = FY^2 + F'Y^2$; therefore $FX.F'X = FY.FY'$; hence it follows that $FX = FY$ and $F'X = F'Y$. The triangles FCX, FCY are therefore equal in all respects, and the angle $FCX = FCY$, that is, the equal semi-diameters CX, CY are equally inclined to the axis.

† Not drawn in figure.

THE PARABOLA.

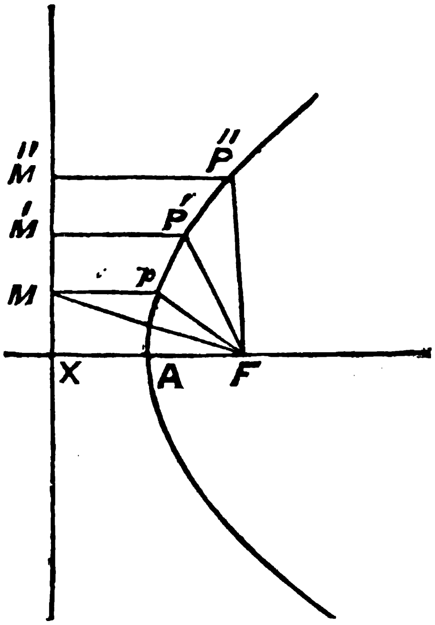
Art. 59.

Construction of the curve :—

Since (by Prop. V.) the distances of any point on a Parabola from the focus and directrix are equal, this may be effected in the following manner :—

Let it be required to find any number of points on a Parabola, of which F shall be the focus and XX′ the directrix. Draw FX perpendicular to XX′ and bisect it in A, then A is the vertex. Take any number of points M, M′, M″, on the directrix, and join FM, FM′, FM″, draw MP, M′P′, M″P″, ... at right angles to XX′, and make the angles MFP, M′FP′, M″FP″ ... equal to the angles FMP, FM′P′, FM″P″.... Then the points P, P′, P″... are on the curve.

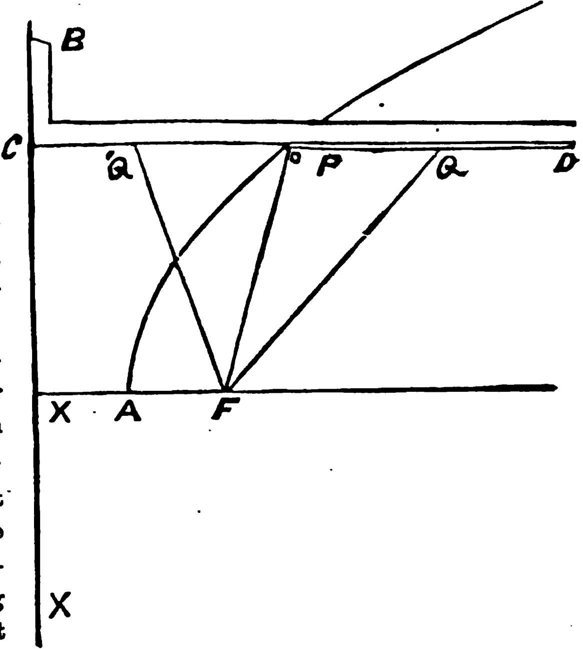
Fig. 51.



Or the construction may be effected Mechanically, thus :—

Let BCD be a rule such that BC is at right angles to CD; then if to the extremity D a string of equal length with DC be fastened, whilst the other end of the string is fixed at F; if the part of the rule BC be made to slide along XX′, whilst the string is kept stretched by a pencil continually in contact with the rule as at P, the point of the pencil will describe the Parabola required. For it is evident that in every position of the rule and pencil that PF = PC.

Fig. 52.



Cor.—The distance of any point in the plane of a Parabola from the focus is less or greater than its distance from the directrix according as the point is within or without the curve. Let Q be within and Q′ without the Parabola, then it is evident (by Eu. i. 20) that $FQ < QP + PF$, that is $< QC$, and $FQ' + Q'P > FP$; therefore $F'Q > FP - Q'P$, that is $> Q'C$.

Art. 60.
Fig. 53.

Let PAp be a Parabola of which F is the focus and XX′ the directrix, PFp a focal chord and PT, pT, tangents at P and p; PM, pm perpendiculars on the directrix, and PN an ordinate to the axis; also join FT. *

By Prop. V. we have the following results :—

(*a*) FP = PM

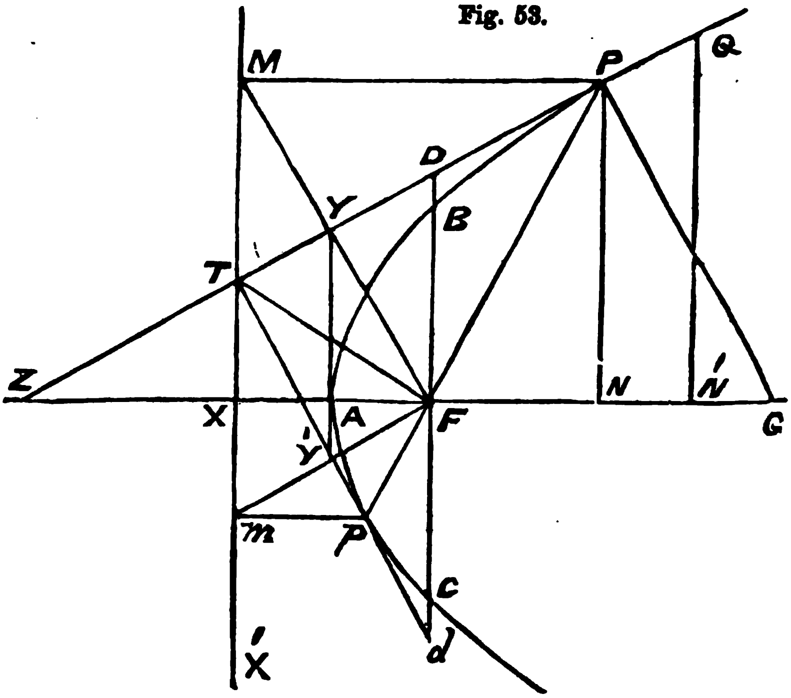
(*b*) $PN^2 = 4AF.AN$

(*c*) angle MPT = FPT

PT being the tangent at any point P.

(*d*) FT is at right angles to PFp and meets the tangents at P, and p, in the same point T on the directrix.

Fig. 53.



Def.—If PZ, PN, PG, be respectively the tangent, ordinate, and normal at a point P of a Parabola, and they meet the axis in the points Z, N, G, then ZN is called the sub-tangent, and NG the sub-normal.

Art. 61.
Fig. 53.

The same construction being made as in the last Art.

Since by (*e*) the angles FPT, MPT are equal, and MPT = FZP, therefore FZ = FP = PM = NX; that is, FZ = NX, but AF = AX; therefore AN = AZ. Therefore the sub-tangent NZ is bisected at the vertex.

Art. 62.
Fig. 53.

Join FM cutting PT in Y. Then since FP = PM, the angles PMY, PFY, are equal, also the angles FPY, MPY, are equal by (*c*), therefore (by Eu. i. 26) MY = FY, and FM is at right angles to PT.

---

* The above construction (Art. 60) or such parts of it as are necessary, is to be understood in each of the Arts. from 60 to 66, and when for the proof of the properties contained in these Articles any additional construction is required it will be stated, and the same thing is generally to be understood in the section on the Parabola, Ellipse, and Hyperbola. The Articles being on the same page with or facing the figures by which they are illustrated, similar or allied properties being as far as possible illustrated by the same figure.

Also if AY be joined, since FY = YM and FA = AX, therefore AY will be parallel to MX, and therefore at right angles to AX.

Further it is evident that the triangles FPY, FYA, are equiangular, for the angle FPY = MPY = FMX = FYA, and the angles FYP, FAY, are right angles therefore the triangles FYP, FAP are similar, and FP : FY = FY : FA; therefore FP.FA = FY^2.

Art. 63.
Fig. 53. By (*d*) the tangents at the extremity of a focal chord meet on the directrix, and conversely, tangents from any point on the directrix touch the curve at the extremities of a focal chord.

Also, since the angles TFP, TMP are equal, being right, and FPT, MPT are also equal, therefore the remaining angles FTP, MTP are equal; that is, PT bisects the angle FTM, similarly pT bisects the angle FTm. Therefore, PT, pT are at right angles; hence, the tangents at the extremities of a focal chord intersect at right angles on the directrix.

We have, therefore, the following rule for drawing tangents to a Parabola from any point T on the directrix. Join TF, and draw PFp at right angles to TF and cutting the curve in P and p. Join PT, pT, these lines are the tangents required.

Art. 64.
Fig. 53. Since the triangles TFP, TMP are equal in all respects, therefore TF = TM; similarly TF = Tm. Therefore a circle with centre T may be described through the points M, F, m; and the angle MFm, being in a semi-circle, will be a right angle.

That is, if perpendiculars be let fall from the extremities of a focal chord on the directrix, the intercept made by them thereon subtends a right angle at the focus.

Art. 65.
Fig. 53. In the Parabola the sub-normal NG is constant; for by the right angled triangle

ZPG we have $PN^2 = ZN.NG = 2AN.NG$ (Art. 61.)
But $PN^2 = 4AF.AN$ by (*b*)
Therefore $4AF.AN = 2NG.AN$
$\therefore$ $2AF = NG$

Therefore the sub-normal NG is constant and equal to the semi Latus Rectum of the Axis.

It is also evident that the normal bisects the supplement of the angle FPM or of its equal PFG.

Draw GL perpendicular to FP, then PL is also equal to the semi Latus Rectum. See Art. 55.

Art. 66.

PROB.—To draw Tangents to a Parabola from a given point O without it.

FIG. 54. With centre O and radius OF, describe a circle cutting the directrix in M and m, through M and m draw MP, mp at right angles to the directrix and meeting the curve in P and p. Join OP, Op, these will be the tangents required.

For the triangles OFP, OMP have their sides equal, therefore (by Eu. i. 8) the angles OPM, OPF are equal, therefore by (*c*) OP is a tangent at P, and similarly Op is also a tangent at P.

Since (by Art. 62) FM, Fm are at right angles to OP, Op, and the tangent at the vertex YAY′ is parallel to the directrix, tangents may also be drawn to the Parabola by the following method. Join OF and on it describe a circle cutting YAY′ in Y and Y′. Join OY, OY′, these lines produced will be the tangents required.

Art. 67.

FIG. 54. Since OM = Om, therefore the angles OMm, OmM are equal; adding to each the right angles mMP, Mmp, the angle OMP = Omp; that is, OFP = OFp.

Hence OF makes equal angles with the focal distances FP, Fp of the points of contact P and p.

Again, draw OD, Od perpendicular to PF, pF. Then since by this Art. the angles OFP, OFp are equal, and the angles at D and d are equal, and the side OF is common to the two triangles OFD, OFd; therefore (by Eu. i. 26) OD = Od.

Art. 68.

FIG. 54. Draw LOl parallel to MXm the directrix. Then since OM = Om and LM = lm, and the angles OML, Oml are equal, for they are the supplements of the equal angles OMP, Omp, (Art. 68) therefore (by Eu. i. 4) OL = Ol.

Art. 69.

Draw OE at right angles to the directrix and produce it to meet the section in Z and Pp in V. Then the angles MOE, mOE are equal.

But $MOE = 2POE + FOE$
And $mOE = 2pOF + FOE$
Therefore $POE = pOF$
Add to each the angle FOE and $pOE = POF$.
That is, OE, OF make equal angles with the tangents.

Art. 70.

FIG. 54. Let PO, pO meet the Axis AA′ in W and S, and join Pp.

Then the angle $PFA' = FWP + FPW = 2OWS$, since $FP = FW$.
Similarly $pFA' = 2FSp = 2OSW$
But $POp = OWS + OSW$
Therefore $PFp = 2POp$, or $POp = \frac{1}{2}PFp$.

Also, since the angles PFp, OFP, OFp are together equal to four right angles, and WOS, POp are equal to two right angles, it follows since PFp has been shown to be double of the angle POp, that OFP, OFp are together double of the external angle WOS.

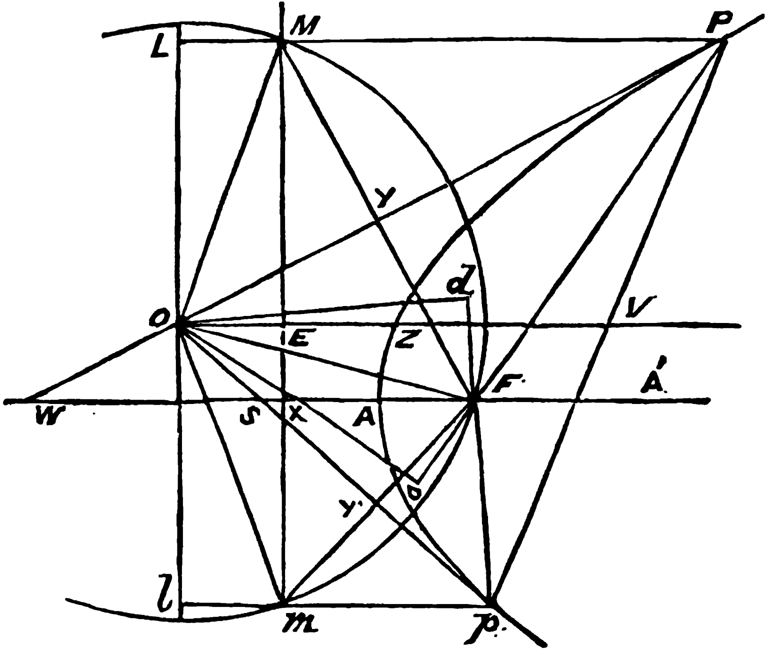
Hence the angle contained by a pair of tangents is equal to half that subtended at the focus by the chord of contact, and the difference between this angle and four right angles is equal to twice the external angle contained by the tangents.

Note.—The angle subtended by the chord of contact and contained by the focal distances of the points of contact is less than two right angles if the focus lies between the chord and the curve, but greater if the contrary be the case.

Def.—A straight line drawn through any point of a Parabola parallel to the axis is called a diameter, and the point through which it is drawn, its vertex.

Art. 71.

Fig. 54.



Since a circle can be described through the points F, M, m, the angle FOp at the centre will equal FMm at the circumference; but FM is at right angles to PO (by Art. 62) therefore the angle FMm = MPO = FPO, and therefore FPO = FOp, and (by Art. 68) the angles OFP, OFp are equal. Therefore the triangles OFP, OFp are similar and

$$PF : OF = OF : Fp$$

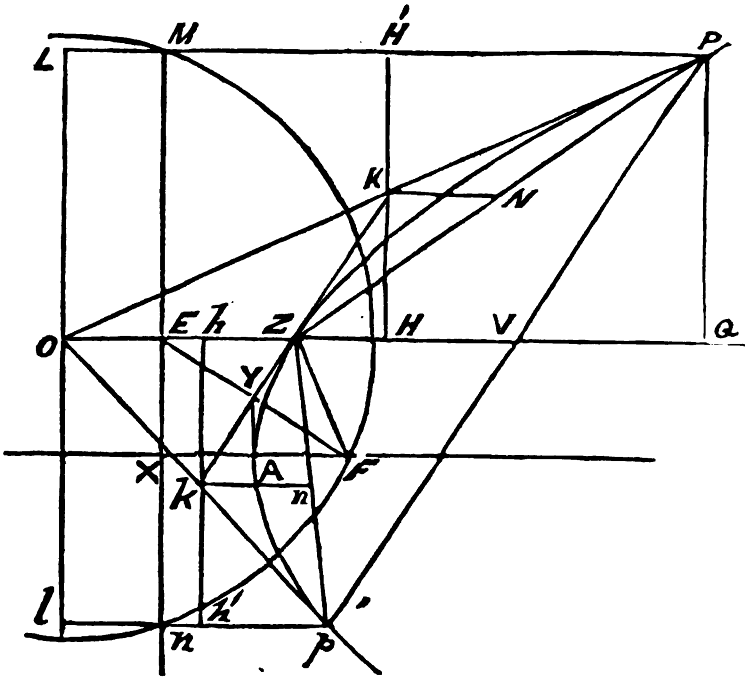
Therefore $FP.Fp = OF^2$

It is plain also from this demonstration that PO makes the same angle with the axis that pO does with OF, the focal distance of O, and similarly the angle pSF = POF.

That is, of two tangents drawn from a given point, either makes the same angle with the axis that the other does with the focal distance of the point from which they are drawn.

The following theorem immediately follows from this Art. :—"If from any points in a given tangent of a Parabola tangents be drawn to the curve, the angles which these tangents make with the focal distances of the points from which they are drawn, are all equal." For by the Art. each of them is, and equal to the angle between the given tangent and the focal distance of its point of contact.

Fig. 55.



Art. 72.
Fig. 55. Having made the same construction (Arts. 67, 71, & 72) through Z draw the tangent KZk meeting OP, Op in K and k, then the tangent KZk is parallel to the ordinates of the diameter OV. Through K and k draw HKH, hkh' parallel to the directrix and meeting OQ, MP, mp in the points H, h,; H', h'. Then in the same manner that OL was shown to be equal to Ol, KH may be shown to be equal to KH' and kh to kh'. Since, therefore, the angle KPH' = KOH, and the angles at H and H' are right angles, therefore (by Eu. i. 26) OK = KP; in the same manner Ok may be shown to be equal to kp; therefore Kk is parallel to Pp; but since OL = Ol (Art. 70) therefore PV = Vp, and therefore Pp is an ordinate to the diameter OV.

We have, therefore, a proof independent of Prop. VIII. that the tangents at the vertices of the diameters of a Parabola are parallel to the ordinates of those diameters.

Hence it appears that the diameter through the point of intersection of a pair of tangents passes through the middle point of the chord of contact, and the tangent at its vertex is parallel to that chord, also the portion of the diameter between the chord of contact and point of intersection of the tangents is bisected by the curve.

Art. 73.
Fig. 55. Let now Kk be any tangent to the Parabola not necessarily parallel to Pp, then if FK be joined, in the same way that the angle FOp was shown to be equal to the angle FPO, the angle FKZ may be shown to be equal to the angle FPO; therefore FKZ = FOp. Hence a circle may be described through the four points O, K, F, k; that is, the circle described about the triangle formed by any three tangents to a Parabola passes through the focus.

Art. 74.
Fig. 55. Draw PQ perpendicular to OV, and let the tangent at A meet the tangent at Z, the vertex of the diameter ZV in Y, and join FY.

Then the angle PVQ = OZY = FZY = FYA (see Arts. 72, 60, & 62) and the angles PQV, FAY are right, therefore the triangles PVQ and FYA are similar, and

$$QP^2 : PV^2 = AF^2 : FY^2 \ (= AF.FZ \text{ by Art. 62.})$$

$$= AF : FZ$$

$$= 4AF.ZV : 4FZ.ZV.$$

But (by Art. 30) $PV^2 = 4FZ.ZV$

Therefore $QP^2 = 4AF.ZV.$

Art. 75.

Fig. 56.

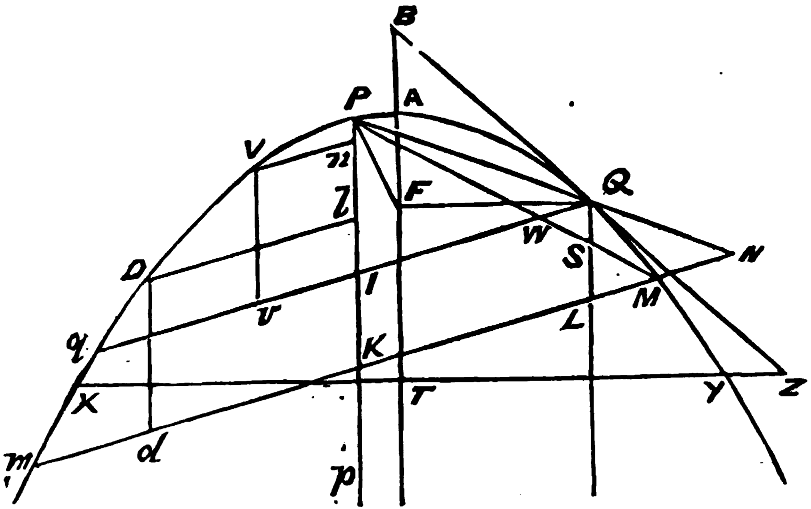


FIG. 56. If a pair of parallel straight lines Qq, Mm, drawn in a Parabola meet with any pair of diameters Dd, Vv in the points d and v, the rectangles of the segments of the parallels are to one another as the abscisses of the diameters, that is,

$$Md.dm : Qv.vq = Dd : Vv.$$

Let Pp be the diameter to which Qq, Mm, are ordinates, and let Qq, Mm meet Pp in the points I, K, and draw the ordinates Vn, Dl to Pp, and join FP, F being the focus.

Then (by Art. 30) $mK^2 = 4FP.PK$, and $Dl^2 = 4FP.Pl$

$qI^2 = 4FP.PI$, and $Vn^2 = 4FP.Pn$

Therefore $Md.dm = mK^2 - Dl^2 = 4FP.Kl$ and $Qv.vq = qI^2 - Vn^2 = 4FP.nI$

Therefore $Md.dm : Qv.vq = 4FP.lk : 4FP.nI$

$= Dd : Vv.$

COR. 1.—When the diameters Dd, Vv coincide with Pp, to which Qq and Mm are ordinates, the above becomes $MK^2 : QI^2 = PK : PI$. That is, the squares of the ordinates of any diameter are to one another as the abscisses they cut off, and therefore the squares of the ordinates of the axis are to one another as the corresponding abscisses, thus (fig 53) $PN^2 : QN'^2 = AN : AN'$. (Compare Art. 30)

COR. 2.—If an ordinate Mm to any diameter Pp meet another diameter Dd in d, and if L equals the parameter of Pp, then $Md.dm = L.Dd$. (See Art. 30.)

Art. 76.

FIG. 56. Let PAQ be a Parabola of which AA′ is the axis and F the focus, draw the focal ordinate FQ to the axis, and the tangent at Q meeting the axis in B, also let XTY be any other ordinate meeting the tangent at Q in Z, then $XZ.ZY = FT^2$.

For by sim. tri. $TZ : TB = QF : FB$; but $QF = FB$; therefore $TZ = TB$;

Also (Art. 11 or 29) $TY^2 = 4AF.AT$.

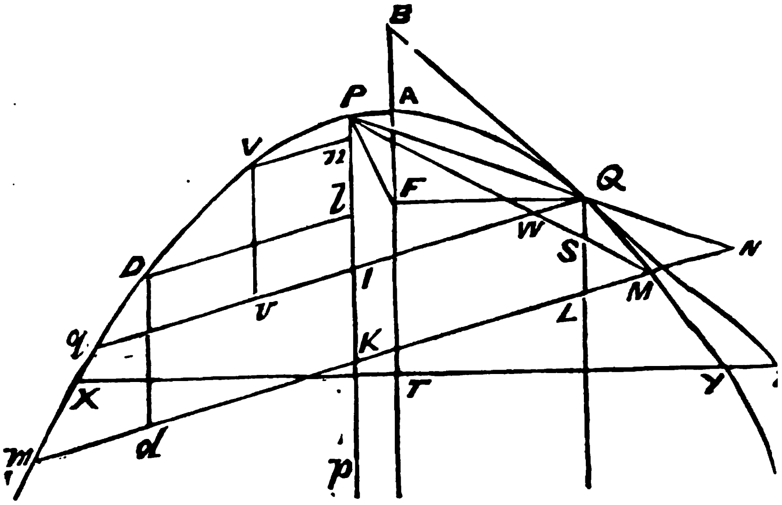
Therefore

$$\begin{aligned} XZ.ZY &= TZ^2 - TY^2 = TB^2 - 4AF.AT \\ &= TF^2 + 2TF.FB + FB^2 - 4AF.AT \\ &= TF^2 + 4AF.AT - 4AF.AT \\ &= TF^2. \end{aligned}$$

Art. 77.

FIG. 56. Again, let P, Q be a any two points on a Parabola and PK, QL diameters, then if from any point M in the Parabola an ordinate KM be drawn to the diameter PK meeting the curve in M, and QL and PQ (produced) in L and N respectively, NK.KL $= KM^2$. Also if PM be joined meeting the diameter QL in S, and QI an ordinate to PK in W, MP.PW $= PS^2$.

Fig. 56.



Draw the ordinate QI which will therefore be parallel to KM.

$\therefore$ by sim. tri. $NK : QI = PK : PI$

But $PK : PI = KM^2 : QI^2$ (by Art. 75)

$\therefore$ $NK : KM = KM : KL$ since $QI = KL$.

Therefore $NK.KL = KM^2$.

Also by parallels $NK : KM = QI : IW$

$= SP : PW$

" " $MK : KL = MP : PS$

Therefore by what has been proved $MP : PS = SP : PW$

Therefore $MP.PW = SP^2$

Art. 78.

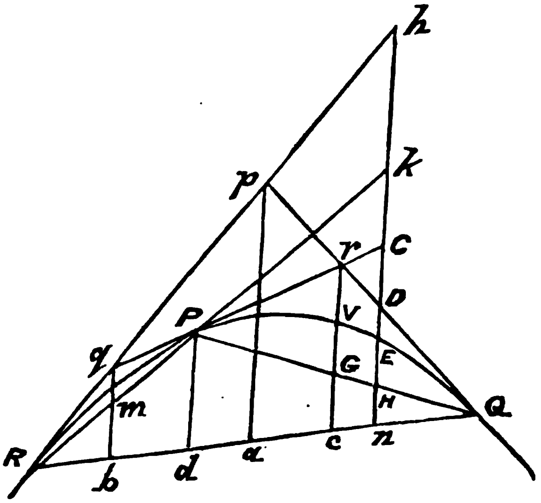
FIG. 57. Let P and Q be any two points on a Parabola, join PQ, and let Pr, Qr be the tangents at P and Q, then if any line be drawn parallel to the axis, and meeting PQ, Pr, Qr and the curve in H, C, D, E, respectively,

$CE : EH = PH : HQ = HE : ED.$

Through r draw the diameter rV meeting PQ in G and the curve in V, and let p denote the parameter of the diameter rV, then since (by Art. 72) $rG = 2VG$, therefore (by Art. 30)

$$2PG^2 = p.rG$$

Fig. 57.



And by sim. tri. $PH : PG = HC : rG$

$\therefore$ $2PH.PG : 2PG^2 = p.HC : p.rG$

But $p.rG$ has been proved equal to $2PG^2$, therefore $p.HC = 2PH.PG = PH.PQ$ since $2PG = PQ$. Also $p.EH = PH.HQ$. (See Art. 75, Cor. 2)

Therefore $HC : EH = PH.PQ : PH.HQ$

$= PQ : HQ$

Div$^{\text{do}}$ $CE : EH = PH : HQ$

In the same manner it may be shown that $HE : DE = PH : HQ$.

And therefore $CE.ED = HE^2$.

Art. 79.

FIG. 57. If from two fixed points R and P on a Parabola, two straight lines RQ, PQ be drawn to any point Q on the curve, and intercepting on a given diameter EH the segments En, EH, respectively, then En shall have to EH a given and constant ratio.

Join RP and produce it to meet the diameter EH (produced) in k, and draw the tangents Rh, PC meeting the same diameter in h and C.

Since then the diameter EH meets the tangents at P and Q in C and D, the chord of contact PQ in H and the curve in E, (by the last Art.) $CE.ED = HE^2$; but the diameter EH also meets the tangents R and Q in h and D, the chord of contact RQ in n, and the curve in E, therefore (by the same Art.) $hE.ED = En^2$.

Therefore $En^2 : HE^2 = hE : CE$

Also (by the last Art.) $hE.EC = Ek^2$

Therefore $hE : EC = Ek^2 : CE^2$

$\therefore$ Ex æquali $En^2 : HE^2 = Ek^2 : CE^2$

And $En : HE = Ek : CE$.

But the points R and P and the diameter are fixed, therefore the points k and C are given, and therefore the ratio of En to HE, which is the same as that of Ek to CE, is given and constant.

Art. 80.

FIG. 57. Let P, Q, R be any three points on a Parabola, and let the tangents at P, Q, R meet each other in r, p, q respectively, then $Rq : qp = qP : Pr = pr : rQ$.

Join RP, PQ, QR and draw diameters through the points p, q, r, P meeting RQ in the points a, b, c, d, and let the diameters through q and r meet RP and PQ in m and G, then (by Art. 72) $Ra = aQ$, $Rm = mP$, $PG = GQ$; and since all diameters are parallel, therefore also $Rb = bd$ and $dc = cQ$; therefore $bc = \frac{1}{2}RQ = aQ$, and taking away the common part ac, $ba = cQ = cd$, and therefore also $bd = ac$, since therefore Rb, bd, ac are all equal, and also ba, dc, cQ,

$Rb : ba = bd : dc = ac : cQ$

$\therefore$ by parallels $Rq : qp = qP : Pr = pr : rQ$.

EXAMPLES ON THE PARABOLA.

No.

1.—Draw a tangent to a Parabola making a given angle with the axis.

2.—The diameter of the circle passing through the Vertex and extremities of the ordinate through the focus, is equal to five times the distance of the focus from the vertex.

3.—In fig. 53, prove that $AY^2 = AF.AN$.

4.—PQ is a common tangent to a Parabola, and the circle described on the Latus Rectum as diameter, prove that FP and FQ make equal angles with the Latus Rectum.

5.—If the normal at P meet the axis in G, and through G a parallel be drawn to the tangent, any straight line drawn through the focus to meet the tangent and the parallel thereto through G, is bisected at the focus.

6.—Given the focus and a tangent, construct the Parabola so that the axis may be parallel to a given line.

7.—Prove that $PY.PT = FZ^2$; also $PY.YT = AF.FP$. (See fig. 53.)

8.—Construct a Parabola with a given focus and two given tangents.

9.—The distance of any point on a Parabola from the focus is equal to the length of the ordinate at the point produced to meet the tangent at the extremity of the Latus Rectum.

10.—If PW be drawn at right angles to AP, meeting the Axis in W, and PN be the ordinate at P, then $NW = 4AF$.

11.—In fig. 53, if NY be joined, prove that $NY = YZ$ and $ZP.ZY = ZF.ZN$.

12.—If FE be the perpendicular from the focus on the normal at P, prove that $FE^2 = PF.AN$.

13.—The locus of the vertices of all Parabolas which have a common focus and tangent is a circle.

14.—If a circle be described about the triangle FPN (fig. 53) the tangent to it from $A = \frac{1}{2}PN$.

15.—If AP be joined and produced to meet QN′ produced in V, (fig. 53) then $PN.N'V = AN' \times$ Latus Rectum.

16.—If QP be joined and produced to meet the axis in W, (fig. 53) then $AW^2 = AN.AN'$.

17.—If PFP′ be a focal chord, and PN, P′N′ the ordinates at P, P′, prove that $AN.N'A = AF^2$.

18.—Show also, that the Latus Rectum is a mean proportional between the double ordinates at P and P′.

19.—The locus of the middle points of focal points of a Parabola is another Parabola, find its Latus Rectum.

20.—If the triangle FPG (fig. 53) be equilateral, then FP is equal to the the Latus Rectum.

21.—If Pp be the chord of contact of tangents from any point O, show that the distance of Pp from F, measured on the axis, is equal to the perpendicular from O to the directrix, but lies on the opposite F to that which O does from the directrix.

22.—If OP, op be a pair of tangents to a Parabola, and OZV be the diameter passing through O and meeting the curve in Z and Pp in V, then F being the focus, if FP, Fp, FZ be joined, prove that FP + Fp = OV + 2FZ; also if KFL be the focal ordinate or parameter of the diameter OV, show that $OP^2 + Op^2 = OV\ (2OV + KL.)$

23.—In fig. 53, if GK be at right angles to FP, then PK = 2AF.

24.—If from any point on a Parabola a tangent be drawn meeting any diameter, and also an ordinate from the point to the same diameter, the segment of the diameter between the ordinate and the tangent will be bisected by the curve.

25.—If AR, ML are any diameters of a Parabola, besides the axis, and AE, MT tangents meeting in Y, prove that the triangle AYT = MYE.

26.—If Pp be a chord perpendicular to the axis, and the perpendicular from p on the tangent at P meets the diameter through P in R, prove that RP is equal to the Latus Rectum, and find the locus of R.

27.—Having given the focus, describe a Parabola through two given points.

28.—If APC be a sector of a circle, of which the radius CA is fixed, and a circle be described touching the radii CA, CP, and the arc AP, the locus of the centre of this circle is a Parabola.

29.—The circle on any focal distance touches the tangent at the vertex, and that on any focal chord touches the directrix.

30.—If the diameter PV meet the directrix in O, and the chord drawn through the focus parallel to the tangent at P in V, prove that VP = PO.

31.—Prove that the locus of the intersection of a diameter PV with the chord drawn through the focus parallel to the tangent at P is a Parabola.

32.—If a circle and Parobola have a common tangent at P, and intersect in Q and R, and QV, UR be drawn parallel to the axis of the Parabola meeting the circle in V and U respectively, then VU is parallel to the tangent at P.

33.—If an ordinate to any diameter pass through the focus, the absciss is equal to one fourth of the parameter of that diameter, and the ordinate is equal to the whole parameter.

34.—If any two diameters of a Parabola be produced to meet a tangent to the curve, the segments of the diameters between their vertices and the tangent are to one another as the squares of the segments of the tangent intercepted between each diameter and the point of contact.

35.—If from a point in a Parabola a perpendicular be drawn to any diameter, and also from the same point an ordinate to that diameter, the square of the perpendicular is equal to the rectangle contained by the abscis of the diameter and the parameter of the axis.

36.—If CD any diameter of a Parabola whose vertex is C, intersect a chord AB in D, and from the ends of the chords straight lines AE, BE be inflected to any point E in the curve cutting the diameter in H and K, H being in AE, and K in BE, then the segments AD, BD of the chord shall have the same ratio as the segments CH, CK of the diameter between its vertex and the inflected lines.

37.—A Parabola being given in position, to find its directrix and focus.

38.—If from a point P of a circle, PC be drawn to the centre, and R be the middle point of the chord PQ drawn parallel to a fixed diameter ACB, then the curve traced out by the intersection of CP and AR is a Parabola.

39.—If two equal tangents OQ, OQ′ be cut by a third tangent, their alternate segments are equal.

40.—If AG be any diameter, AB a tangent, AH perpendicular to the axis, and BD parallel to the axis, meeting the tangent in B and curve in D, then $AH^2 = BD \times$ Latus Rectum.

41.—If EG, HQ be ordinates to the axis AQ, BET a tangent at E, EL a parallel to AQ meeting BQ in L, and QH be produced to meet the tangent in B, then BH : HL = HL : EM.

42.—If AD, PQ be two diameters, AC a tangent at A intersecting QP in C, and if any point E be taken in AD, and CL be made equal to AE, and the lines PE, AL be drawn, they will intersect at S a point on the curve.

43.—If DM, DN be two tangents, MN the lines joining the points of contact, and if any line GL be drawn parallel to either tangent DN cutting the curve in R and L, MN in H, and DM in G, then GR, GH, GL are continually proportional.

44.—If from the focus F of a Parabola FY, FZ be perpendiculars to the tangent and normal at any point, YZ is parallel to the axis.

45.—Prove that the locus of the foot of the perpendicular from the focus on the normal is a Parabola.

46.—If R be the centre of the circle described about the triangle POp, (fig. 54) prove that the circle described about PRp will pass through the focus.

47.—PFp is any focal chord of a Parabola, prove that AP, Ap will meet the Latus Rectum in two points Q and q, whose distances from the focus are equal to the ordinates of P and p.

48.—If the extremities of a focal chord be joined with any point on the curve, and the joining lines be produced to meet the directrix in D, d, then half the Latus Rectum is a mean proportional between XD, Xd—X being the foot of the directrix.

49.—PG, pg are normals at the extremities of a focal chord, prove that FG.Fg = AF.Pp.

50.—If a Parabola roll upon another equal Parabola the focus traces out the directrix, is any limitation required ?

51.—The squares of the normals at the extremities of a focal chord are together equal to the square of twice the normal perpendicular to the chord.

52.—From the foot of the directrix a chord is drawn to a Parabola, prove that the ordinates of the points in which the chord meets the Parabola contain a rectangle equal to the square of the Latus Rectum.

53.—Describe a Parabola touching four given straight lines.

54.—If the diameter PV meet the directrix in Q and the focal chord parallel to the tangent at P in V, prove that $PV = PQ$.

55.—G is the foot of the normal at a point P of the Parabola, Q is the middle point of FG, and X is the foot of the directrix, prove that $QX^2 - QP^2 = 4AF^2$.

56.—Having given the vertex of a diameter and an ordinate thereto, construct the Parabola.

57.—If the tangent and normal at a point P of a Parabola meet the tangent at the vertex (A) in K and L respectively, prove that $KL^2 : FP^2 = FP - AF : AF$.

58.—If the ordinate of a point P bisects the sub-normal of a point P′, prove that the ordinate of P is equal to the normal of P′.

59.—QVQ′ is an ordinate of a diameter PV, and any chord PR meets QQ′ in N, and the diameter through Q in L, prove that $PL^2 = PN.PR$.

60.—If AP, AQ be two chords drawn from the vertex at right angles to each other, and PN, QM be ordinates, the Latus Rectum is a mean proportional between AN and AM.

61.—If from any point P of a Parabola, two straight lines PE, PH be drawn making equal angles with the normal at P, then $FG^2 = FE.FH$.

62.—All chords which subtend right angles at the vertex A pass through a fixed point on the axis.

63.—A system of parallel chords is drawn in a Parabola, prove that the locus of the point which divides each chord into segments containing a constant rectangle is a Parabola.

64.—The area of the triangle formed by three tangents to a Parabola is equal to half the area of the triangle formed by joining the points of contact.

65.—If the normal at a point P of a Parabola meet the curve in Q, and the tangents at P and Q intersect in T, prove that T and P are equidistant from the directrix.

66.—If QR be a double ordinate, and PD a straight line drawn parallel to the axis from any point P of the curve and meeting QR in D, prove that $QD, RD = 4AF.PD$.

67.—Inscribe in a given Parabola a triangle having its sides parallel to three given straight lines.

68.—PNP′ is a double ordinate, and through R, a point in the tangent at P, RQM is drawn perpendicular to PP′ and meeting the curve in Q, prove that QM : QR = P′M : PM.

69.—A circle passing through the focus cuts the Parabola in two points, prove that the angle between the tangents to the circle at those points is four times the angle between the tangents to the Parabola from the same points.

70.—If a circle cut a Parabola in four points, two on one side of the axis, and two on the other, the sum of the ordinates of the first two is equal to the sum of the other two points. Extend this theorem to the case in which three of the points are on one side of the axis and one on the other.

71.—If AQ be any diameter of a Parabola, and any line CE be drawn cutting it in C, and the curve in D and E, then if the ordinates (to AQ) DB, EQ be drawn, $CA^2 = AB.AQ$.

72.—If a triangle be inscribed in a Parabola, the points where the sides produced meet the tangents at the opposite angles are in the same straight line.

73.—The focus of a Parabola is joined with each of the angles of a circumscribing triangle, prove that the perpendiculars to the joining lines through the angles all meet in one point.

74.—If a Parabola be inscribed in a triangle, the directrix passes through the point of intersection of the perpendiculars drawn from the angular points of the triangle to the opposite sides.

75.—If from any point Q in the line BQ, which is perpendicular to the axis CAB of a Parabola, vertex A, QR is drawn parallel to the axis to meet the curve in R, prove that if CA be equal to AB, the AQ, CR will meet on the Parabola.

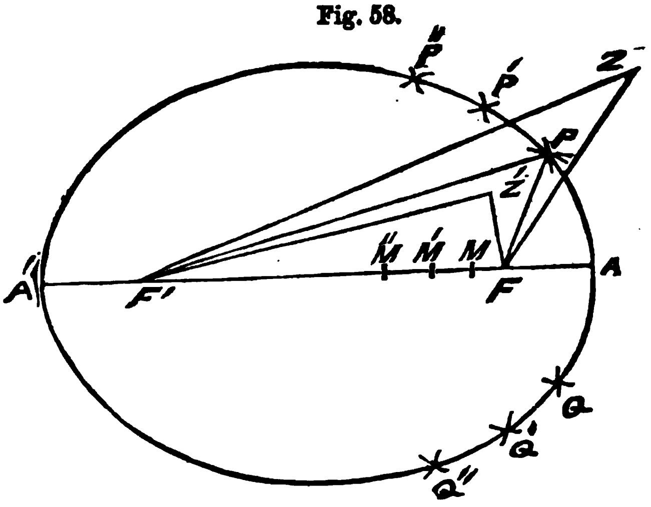
76.—The squares of the tangents from any point to a Parabola are to one another as the focal distances of the points of contact.

77.—Prove the following construction for drawing tangents to a Parabola from a given point O :—Draw the diameter through O meeting the curve in Z and the directrix in R, take ZN equal to ZR and ZV equal to ZO, join FN, and draw PVp parallel to FN, then if OP, Op be joined, these are the tangents required.

THE ELLIPSE.

Art. 81.

Fig. 58.



Construction of the curve.

Fig. 58. Let it be required to find any number of points of an Ellipse of which AA′ is the axis and F and F′ the foci.

Between F and F′ take any number of points M, M′, M″ .... with centre F and distances AM, AM′, AM″ .... describe circles, and with centre F′ and distances A′M, A′M′, A′M″ .... describe circles cutting the former in the points P, P′, P″ .... Q, Q′ Q″ ...... The curve drawn through the points P P′ P″ .... Q, Q′, Q″ will be the Ellipse required.

For it is evident that the sum of the distances of any of the points P, P′ P″ ..... Q, Q′, Q″ .... from the foci F and F′ is constant and always equal to AA′, which is a property of the Ellipse demonstrated in Prop. V.

The Ellipse may also be constructed mechanically, thus :—

Fasten a piece of thread at the points F and F′ by two pins, and keeping the thread stretched by means of a pencil at P, trace out the curve which will be the Ellipse required. The length of the thread must be the same as that of AA′, the axis of the curve.

Cor.—The sum of the distances of any point in the plane of an Ellipse from the foci is greater or less than the axis according as the point is without or within the curve.

Let Z be without and Z′ within the curve.

For (by Eu. i. 21) $FZ + F'Z > FP + F'P$ and $\therefore > AA'$

And $FZ' + F'Z' < FP + F'P$ and $\therefore < AA'$.

Art. 82.
Fig. 59. Let ABA'B' be an Ellipse of which AA' and BB' are the axes major and minor (Art. 26) and F, F' the foci, XY a directrix, and C the centre.

Join FB, F'B, then since FC = F'C and BC is common, and the angles BCF, BCF' are right, therefore (by Eu. i. 4) FB = F'B; but FB + F'B = AA' = 2AC, therefore FB = F'B = AC; also $BC^2 = BF^2 - FC^2 = AC^2 - FC^2$ = AF.FA'.

Art. 83.
Fig. 59. Since by Prop. V. FA : AX = FA' : A'X, therefore AA' is harmonically divided in F and X, and it is bisected in C; therefore CA^2 = CX.CF, (See Art. 15)

It is also evident that if tangents be drawn to the Ellipse from X the foot of the directrix, that they will touch the curve at the extremities of the Latus Rectum, or line DFd drawn through the focus at right angles to the axis AA'. For by Prop. V, if DFd be drawn at right angles to the axis, the tangents at D and d meet the directrix in the point X, that is, in the point in which the perpendicular through F to Dd meets XY.

Art. 84.
Fig. 59. The results of Prop. V. for the Ellipse are as follows :—

(*a*) If ZPW be a tangent at any point P on the curve, the angle FPZ = F'PW.

(*b*) FP + F'P = AA', P being any point on the Curve.

(*c*) FT perpendicular to the focal chords PFp meets the directrix in the same point as the tangents at P and p.

(*d*) If PM be a perpendicular on the directrix,

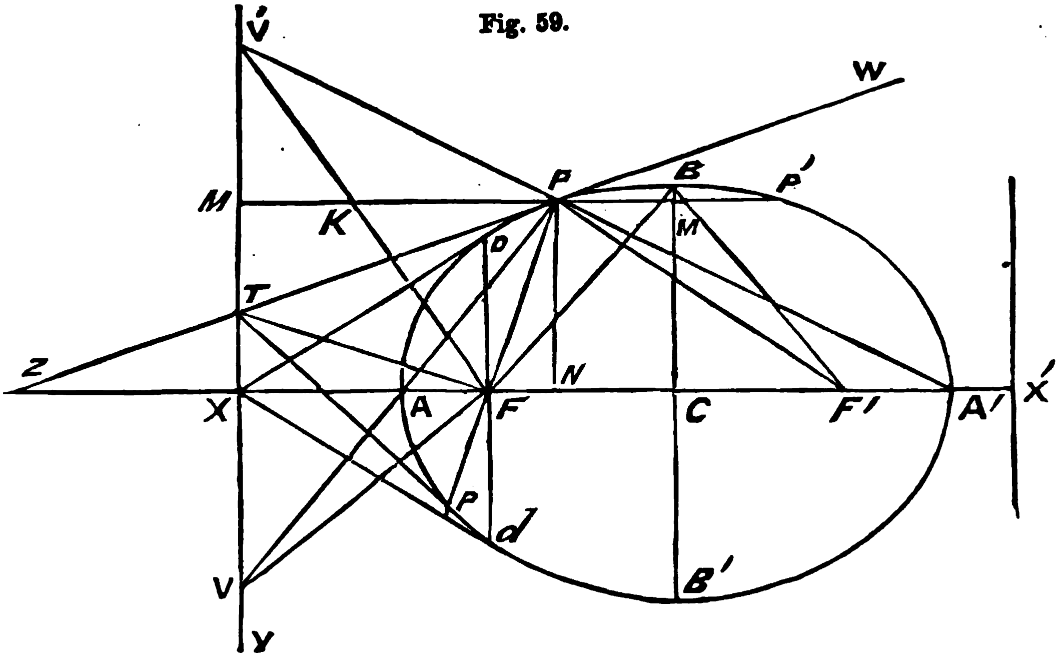
FP : PM = AA' : XX' = AF : AX
= A'F : A'X.

Art. 85.
Fig. 59. Join AP, A'P, and produce them to meet the directrix XY in V and V', then VV' subtends a right angle at the focus F.

Join FV, FV', and draw PM at right angles to the directrix and meeting FV' in K
Then (by *d*, Art. 80) FP : PM = FA' : A'X = PK : PM by sim. tri.
Therefore FP = PK.

Hence the angles PKF, PFK are equal, but PKF = KFX, therefore PFK = KFX; that is, FV' bisects the angle PFX. Similarly if VF be produced to meet MP, it may be shown that VF bisects the supplement of PFX; therefore VF, V'F are at right angles. That is, VV' subtends a right angle at the focus; and therefore, since VFV' is a right angled triangle and FX is perpendicular to VV', VX.XV' = FX^2.

Fig. 59.



Art. 86. Draw the ordinates PN, PM to the axis AA′ and BB′ respectively.

Fig. 59. Then by Prop. VII. $PN^2 : AN.NA' = BC^2 : AC^2$

Similarly $PM^2 : BM.MB' = AC^2 : BC^2$.

The first of these proportions may be proved otherwise, thus :—

By sim. tri. $PN : AN = VX : AX$

„ „ $PN : A'N = V'X : A'X$

Therefore $PN^2 : AN.NA' = FX^2 \ (= VX.XV') : AX.XA'$.

Hence when P coincides with B

$$BC^2 : AC^2 = FX^2 : AX.XA' = PN^2 : AN.NA'$$

Again since (Art. 83) $XC.CF = AC^2$ and $AC^2 - FC^2 = BC^2$

Therefore $XC.CF - CF^2 = XF.FC = BC^2$

∴ $XC : XF = AC^2 : BC^2 = AN.NA' : PN^2$

Art. 87. In Art. 56 it was shown that the rectangles under the segments of any two focal

Fig. 59. chords are to one another as the chords themselves. If therefore Pp be any focal chord, and QCq the diameter parallel thereto $AA' : Pp = AF.FA' : PF.Fp$.

But by Prop. VII $AF.FA' : PF.Fp = CA^2 : CQ^2$

$= AA'^2 : Qq^2$

∴ $AA' : Pp = AA'^2 : Qq^2$

∴ $AA' : Qq = Qq : Pp$

And $AA'.Pp = Qq^2$.

Def.—The director circle of an Ellipse is the circle described from one of the foci as centre, with radius equal to the axis of the curve.

Art. 88.

PROB.—To draw tangents to an Ellipse from a given point P without it.

Fig. 60. Let AA′ be the axis, F and F′ the foci, and C the centre of the Ellipse.

First Method.—With centre F′ and radius AA′, describe the director circle, and with centre P and radius PF, describe a circle cutting the director circle in the points N and n. Join F′N, F′n meeting the Ellipse in the points T, t; PT, Pt joined are the tangents required.

For since F′N = AA′ = FT + F′T ∴ TN = TF and PF = PN,* therefore the triangles PFT, PNT have their three sides equal, and therefore PT makes equal angles with the focal distances of P, and is therefore a tangent at P, (by Art. 80) similarly Pt is a tangent at t.

Second Method.—With centre C and radius CA, describe the auxiliary circle. Join PF and bisect it in O; and with radius PO, describe a circle cutting the auxiliary circle in two points Y and Z; PY, PZ joined and produced are the tangents required.

Join FY and produce it to N so that YN = FY, and join F′N meeting PY in T, also join YC, F′T.

Then since FY = YN, and YT is common, FY, YT = NY, YT, and the angle FYT = NYT, being right angles, (Eu. iii. 31) therefore (Eu. i. 4) FT = TN, and the angle FTY = NTY = PTF′, but since FC = CF′, and NY = YF, therefore F′N is parallel to and equal to 2CY, that is to AA′; therefore FT + F′T = F′N = AA′; therefore the point T is on the Ellipse, and PT is a tangent since it has been shown that PT makes equal angles with the focal distances of P. Similarly PZ (produced) may be shown to be a tangent at t.

These Methods for drawing tangents to an Ellipse may be demonstrated also as follows :—

With centre C and radius CA, describe the auxiliary circle. Join P with one of the foci F, and bisect PF in O. With centre O and radius OF, describe a circle cutting the auxilarly circle in Y and Z; also with centre at the other focus F′ and radius AA′, describe the director circle; and with centre P and radius PF, describe a circle cutting the director circle in two points.

Join FY, FZ, and produce them to meet the circle described from P, with radius PF in N and n. Join F′N, F′n meeting PY, PZ in T and t, then the points T and t are on the curve, and PT, Pt are the tangents required.

Join FT, Ft, CY, CZ, then since PY is perpendicular to FN, (for the angle PYF is in a semi-circle) therefore (by Eu. iii. 3) FY = YN. Therefore FY, YT = NY, YT, and they contain equal angles, and therefore (by Eu. i. 4) FT = NT, and the angle FTY = NTY; wherefore FT + F′T = F′N. But since FY = YN and FO = CF′, therefore F′N = 2CY = AA′, therefore FT + F′T = AA′, hence the point T is on the Ellipse, and the angles FTY, F′TY have been shown to be equal; therefore PT is a tangent at T, and similarly it may be shown that Pt is a tangent at t.

---

* PN is omitted in the figure.

In the course of this proof it has been shown that FN = AA′, therefore the point N, and also n, are on the director circle, being the intersections of that circle with the circle described from P with radius PF.

Therefore either the intersections of the circle O with the auxiliary circle, or the intersections of the circle P with the director circle, are sufficient practically to determine the position of the tangents, for in the one case if PY, PZ be joined, they are the tangents, and in the other case if FN, Fn be joined, these lines intersect the curve in the points of contact.

Cor.—From the preceeding demonstration it appears that the perpendiculars from the foci on the tangents meet them on the auxiliary circle; and if Y, Y′, Z, Z′ be the feet of the perpendiculars, and CY, CY′, CZ, CZ′ be joined, they are each equal to AC.

Art. 89.
Fig. 60.

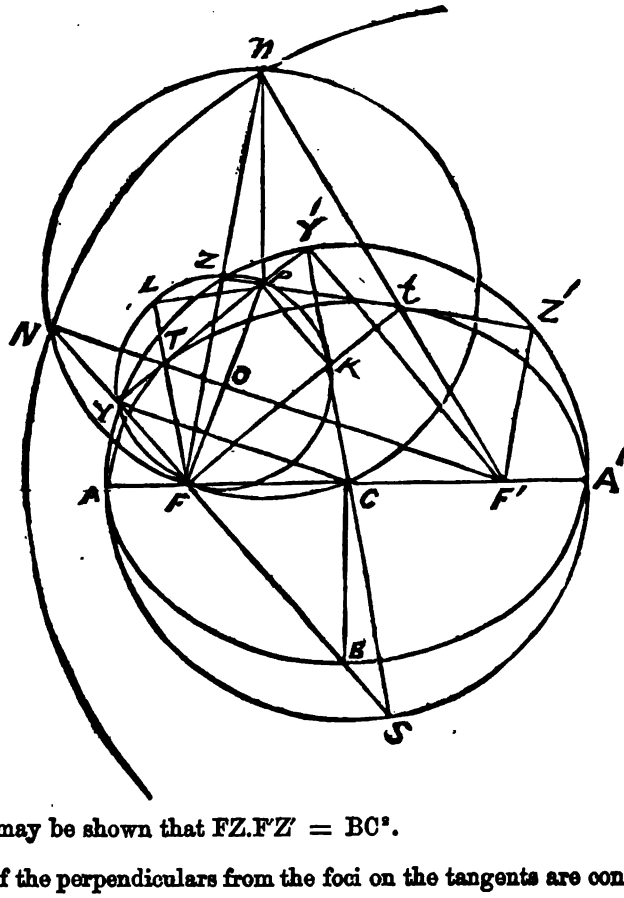
Let FY, F′Y′, FZ, F′Z′ be the perpendiculars on the tangents from the Foci. Join Y′C and produce it to meet the auxiliarly circle again in S, and join FS.

Then since FC, CS are equal to F′C, CY′, and they contain equal angles, therefore (by Eu. i. 4) FS = F′Y′, and the angle CFS is equal to the angle CF′Y′, that is to the angle YFA, therefore YFS is a straight line, and therefore (by Eu. iii. 36) YF.FS = AF.FA′.

But AF.FA′ = BC^2 (by Art. 78.)

Therefore FY.F′Y′ = BC^2.

Fig. 60.



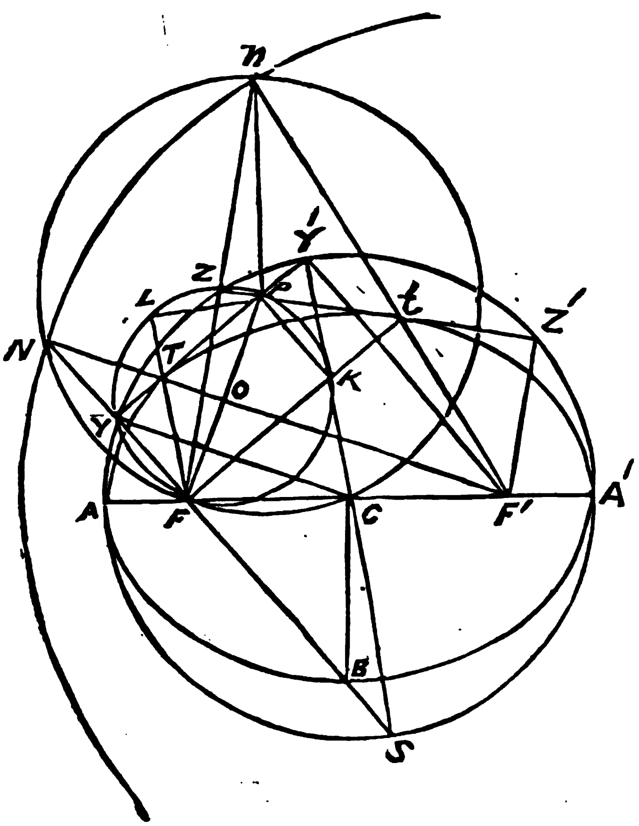
In the same manner it may be shown that FZ.F′Z′ = BC^2.

That is the rectangles of the perpendiculars from the foci on the tangents are constant and equal to the square of the semi-axis minor.

Art. 90.
Fig. 60. Join YZ, Y'Z',* then since

FY.F'Y' = FZ.F'Z'
YF : FZ = F'Z' : F'Y'

Fig. 60.



But since the angles at the points Y, Y' Z, Z' are right, the figures YFPZ, Y'PF'Z' can be inscribed in circles, therefore the angles YFZ, Y'F'Z' are equal to the angle ZPY, and therefore to one another; wherefore (by Eu. vi. 6) the triangles FZY, F'Z'Y' are equangular, and the angle FZY = F'Y'Z'; that is FPY = F'PZ'* .Hence the tangents make equal angles with the focal distances of the point from which they are drawn.

It is also evident that YP.PY' = ZP.PZ', and the triangles YPZ, Z'PY' are similar; and that the angle ZFP = Y'F'P, also the angle YFP = Z'F'P, for they are the supplements of the equal angles YZP, Z'Y'P.

Art. 91.
Fig. 60. The tangents PT, Pt subtend equal angles at the foci.

For joining PN, pn, the triangles PnF', PNF' being equal in all respects, the angle PnF = PNF' But the triangles PFt, Pnt are in all respects equal, therefore the angle Pnt = PFt; and similarly from the equal triangles PNT, PTF, the angles PNT, PFT are equal; therefore the angle PFt = PFT, that is, the tangents PT, Pt subtend equal angles at the focus F, and in the same manner they subtend equal angles at the focus F'.

Cor.—If FT, Ft meet the circle on FP in L and K respectively, and PL, PK be joined, they will be equal since they subtend equal angles at the point F, and each of the angles PLF, PKF being in a semi-circle is a right angle, therefore PL, PK are perpendicular to FT, Ft. Hence the perpendiculars drawn from the intersection of a pair of tangents to the lines joining either of the foci with the points of contact are equal.

---

* Omitted in figure as also F'P.

Art. 92.

Fig. 61.

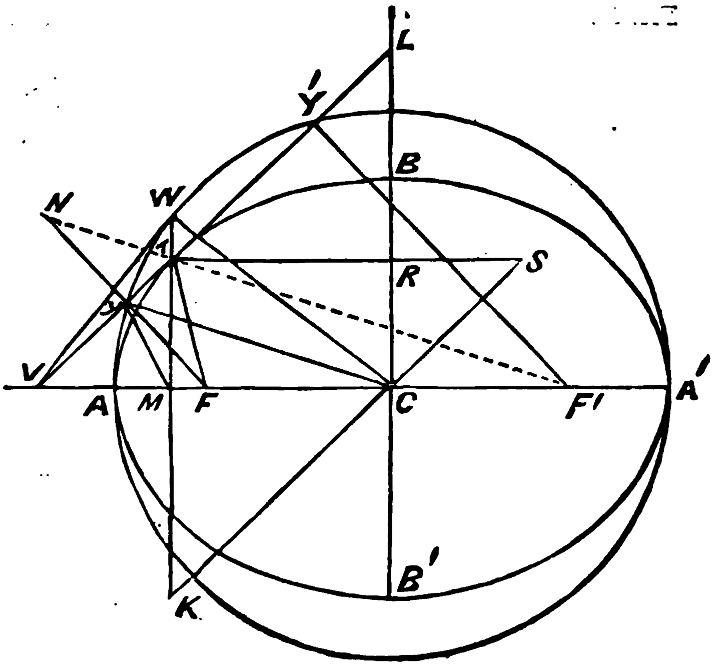


Fig. 61. Let ABA′B′ be an Ellipse of which AA and BB′ are the axis, F and F′ the foci, and C the centre; and let the tangent at any point T on the Ellipse meet the major and minor axes in V and L respectively. Draw the ordinate TM to the axis and produce it to meet the auxiliary circle in W, and join YM, YC, FY, FT, FT, and produce the latter to to meet FY produced in N.

Then since (by Art. 87) CY is parallel to FN the angle CYF = TNF = TFN, and the angle TFN = TMY, for a circle may be described about the figure TYMF, therefore the angle CYF = TMY; add to each the right angle VYF or TMC, therefore the angle VYC = YMC, and therefore the triangles VYC, YMC are similar, wherefore VC : CY = CY : CM.

Therefore $$VC.CM = CY^2 = CW^2.$$

If therefore VW be joined it will be a tangent to the circle. Hence if from any point on the axis tangents be drawn both to the Ellipse and its auxiliary circle, the line joining the points of contact will be an ordinate to both curves. Hence, also VM.MC = AM.MA, for each is equal to WM^2.

Art. 93.

Fig. 61. Draw TR parallel to AA′ and meeting BC in R, then

By sim. tri. $LC : TM = CV : MV$

Therefore $LC.CR : TM^2 = CV.CM : MV.CM$

And alternately $LC.CR : CA^2 (= CV.CM) = TM^2 : AM.MA$ (by Art. 92)

Art. 86 $= BC^2 : CA^2$

Therefore $LC.CR = BC^2$.

Art. 94.

Fig. 62. In like manner it may be shown that if pCP be any diameter of an Ellipse, and QN an ordinate thereto, if the tangent QT be drawn meeting pP produced in T, that TC. CN $= CP^2$.

For draw the tangents PL, pK meeting QT in L and K.

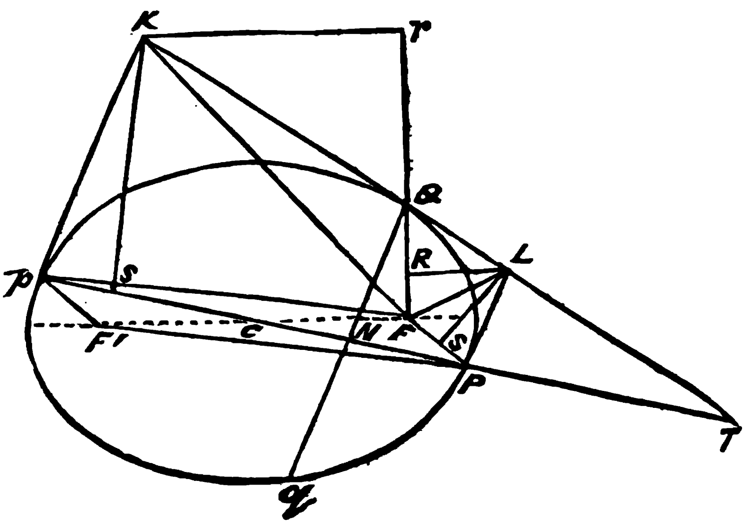


Fig. 62.

Then by Prop. VII.
$PL^2 : QL^2 = pK^2 : KQ^2$
And alternately
$PL^2 : pK^2 = QL^2 : KQ^2$
But by sim. tri. (Art. 23) $PL^2 : pK^2 = LT^2 : TK^2$
Therefore ex æquali $LT : TK = QL : QK$
And by parallels $PT : Tp = PN : pN$.

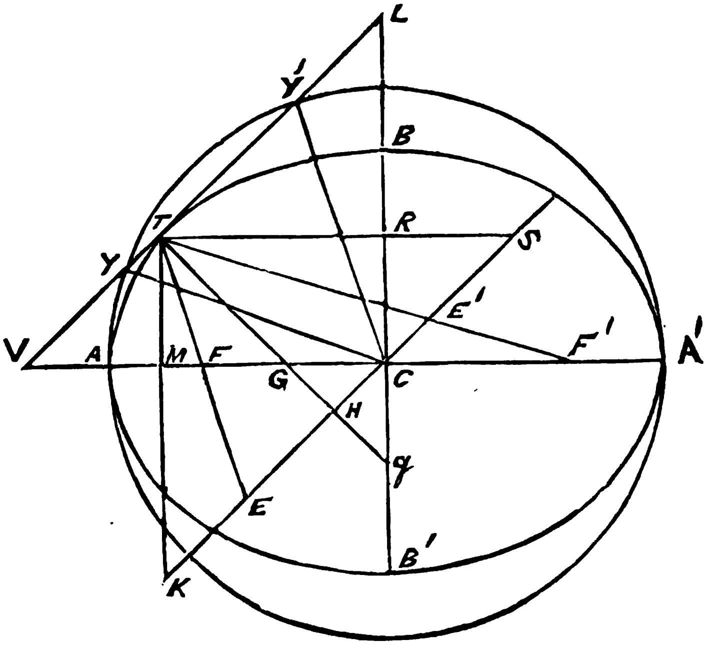
Therefore pP is harmonically divided in N and T, and it is bisected in C, therefore $TC.CN = CP^2$

This may be proved otherwise as follows :— (fig. 62)

F being a focus, join FQ, FP, Fp, and draw LR, LS, Kr, Ks perpendiculars on FQ, FP, FQ, (produced) and Fp respectively, and join KF, LF ; then (by Art. 91) the angles LFP, LFQ are equal, and the angles LRF, LSF are right, therefore (by Eu. i. 26) LS = LR ; also (by Art. 91) the angles KFp, KFQ are equal, and the angles KrF, KsF are right, therefore again (by Eu. i. 26) Kr = Ks. (See Cor., Art. 91)

Also it is evident that the angles FPL, FpK are equal, for if FP be joined (by Prop. V) FP and F'P make equal angles with the tangent PL, but F'P makes the same angle with PL that Fp does with pK, therefore the angles FPL, FpK are equal, and the angles at S and s are right, therefore the triangles LPS, Kps are similar, and $LP : Kp = LS : Ks$
$= LR : Kr$
But by sim. tri. $LP : Kp = TL : TK$
Also $LR : Kr = LQ : QK$
$\therefore$ $LT : TK = LQ : QK$
And by parallels $PT : Tp = PN : Np$

Fig. 63.



Art. 95.

Fig. 63. Through C the centre of the Ellipse ABA′B′ draw ECE′ parallel to the tangent VT and meeting TF, TF′ in E and E′, then TE = TE′ = AC, also FE = F′E′

Let YY′ be the intersections of the tangent with the auxiliary circle, and join CY, CY′, then since CY′ is parallel to F′T and CE to TVY′, therefore TE = CY′ = AC; similarly TE′ = CY = AC.

And therefore $\quad TE = TE' = AC$

Also since $\quad FT + F'T = 2AC = TE + TE'$

Therefore $\quad TE = TF + FE'$

And consequently $\quad FE = F'E'$.

Art. 96.

Fig. 63. Draw the normal TGg meeting the axes in G and g and CE in H, also let CE meet TM produced in K, and draw TR parallel to the axis AA′.

Then since the figure MGHK may be inscribed in a circle, the rectangle $HT.TG = KT.TM = LC.CR = BC^2$.

Similarly if TR, CE′ be produced to meet in S, a circle may be described about the figure SRHg.

Therefore $\quad HT.Tg = ST.TR = VC.CM = AC^2$

Again, since TgR, TMG are similar triangles

$$\begin{aligned} MG : MC\ (= TR) &= TG : Tg \\ &= TH.TG : TH.Tg \\ &= BC^2 : AC^2 \end{aligned}$$

Also since $\quad MC : MG = AC^2 : BC^2$

Div^{do} $\quad$ $$\begin{aligned} MC : CG &= AC^2 : AC^2 - BC^2 \\ &= AC^2 : FC^2 \end{aligned}$$

CONJUGATE DIAMETERS.

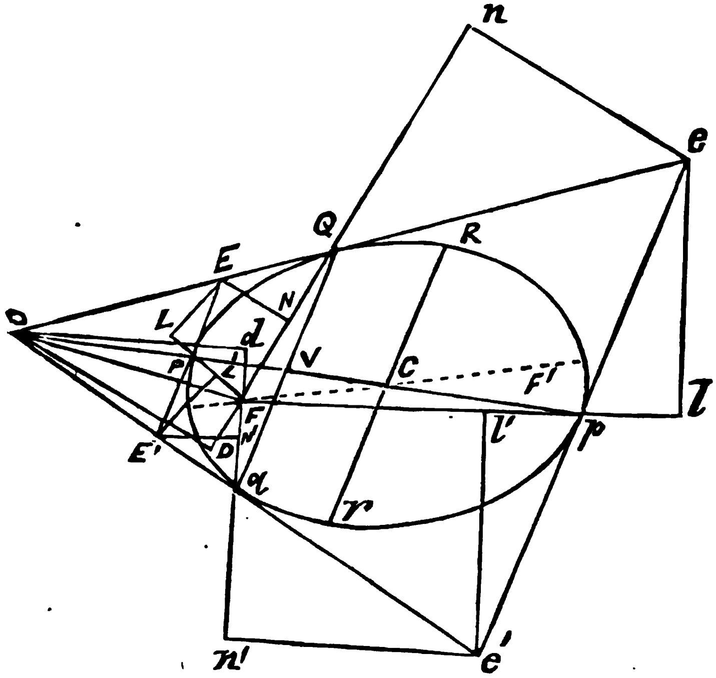
Art. 97.

It has been shown in Prop. VIII. that in any Conic section the diameters bisect all chords drawn in the section parallel to the tangents at their extremities.

In the case of the Parabola an independent proof of this has been given in Art. 72.

The following is the analogous proof in the case of the Ellipse.

Fig. 64.



Let PCp be any diameter of an Ellipse of which F and F' are the foci, and C the centre. Draw the tangents EPE', epe', which (by Art. 23) will be parallel; also draw any chord QVq parallel to EE' or ee', and meeting Pp in V, then Qq is bisected in V.

Draw the tangents OQ, Oq to the Ellipse meeting the parallel tangents EE' ee' in the points E, E', e, e' . Join QF, Fq, and draw OD, EN, en; od, E'N', e'n' perpendiculars thereto; also join FP, Fp, and let fall the perpendiculars EL, E'L'; el, e'l' upon them. Then (by Art. 91) OD = Od, EN = EL, E'N' = E'L', en = el, and e'n' = e'l'

But by sim. tri. $\quad OD : EN = OE : EQ$

" " $\quad = OE' : E'q$

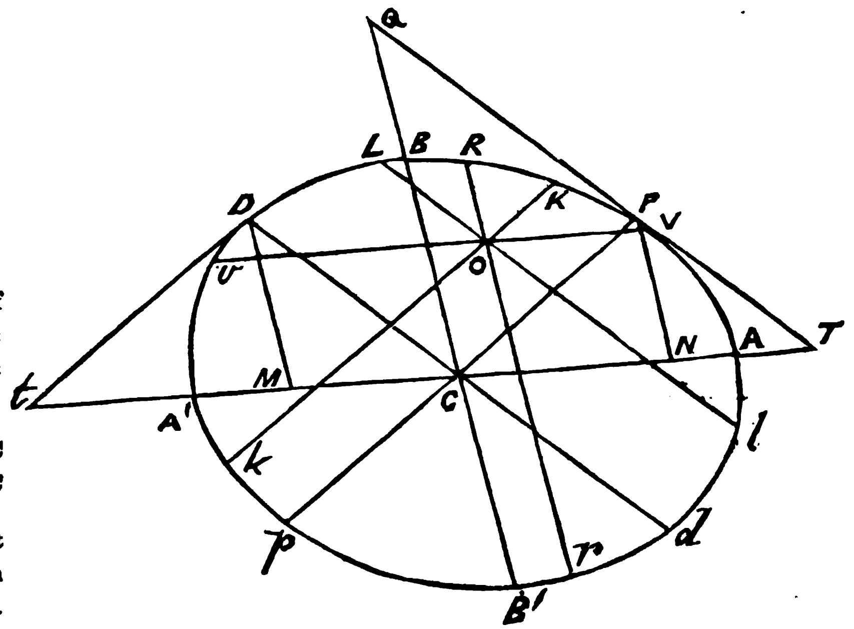
" " $\quad = Od : E'N'.$

Therefore EN = E'N' and ∴ EL = E'L', and consequently (by Eu. i. 26) EP = E'P; in the same way ep may be shown to be equal to e'p.

Therefore Pp must pass through O, and therefore bisects Qq, which is parallel to EE' or ee', in the point V.

If RCr be the diameter parallel to Qq or EE', Rr, as already defined (Art. 25) is said to be conjugate to PCp, and by Prop. VII. $QV^2 : PV.Vp = CR^2 : CP^2$.

Fig. 65.



Art. 98.

If ACA′, BCB′ be a pair of conjugate diameters, and PCp, DCd, another pair; if PN, DM be ordinates of ACA′, then $CN^2 = AM.MA'$ and $CM^2 = AN.NA'$; also

$CM : PN = AC : BC$
$DM : CN = BC : AC$

Draw the tangents at P and D and let them meet ACA′ in T and t.

Then (by Art. 88) $CN.CT = AC^2 = CM.Ct$
Therefore by sim. tri. $CM : CN = CT : Ct$
" " $= PT : CD$
" " $= PN : DM$
" " $= CN : Mt$
Therefore $CN^2 = CM.Mt = CM.Ct - CM^2 = AC^2 - CM^2 = AM.MA'$
Similarly $CM^2 = AN.NA'$
Also since $DM^2 : AM.MA' = BC^2 : AC^2$
Therefore $DM : CN = BC : AC$
Similary $CM : PN = AC : BC$
∴ Ex æquali $DM : CN = PN : CM$
And $DM.CM = CN.PN.$

Art. 99.

Fig. 65. In the course of the above proof it is also shown that $CN^2 + CM^2 = AC^2$

In the same manner, since PN, DM are parallel to BC, $PN^2 + DM^2 = BC^2$

These results being true for any pairs of conjugate diameters including the axes—if AA′, BB′ are the axes, then since they are at right angles, it follows from the last two results by addition that

$$CN^2 + CM^2 + PN^2 + DM^2 = AC^2 \;\; BC^2$$

That is $$CP^2 + CD^2 = AC^2 + BC^2$$

Or the sum of the squares of any pair of conjugate diameters is equal to the sum of the squares of the axes.

Art. 100.

Fig. 65. Produce the tangent at P to meet BC produced in Q.

Then by sim. tri, $PT : PN = CD : DM$
Also $QP : CN = CD : MC$
Therefore $TP.PQ : CN.NP = CD^2 : DM.MC$

But $CN.NP = DM.MC$; therefore $TP.PQ = CD^2$
That is, the rectangle of the segments of a tangent to an Ellipse between the point of contact and any pair of conjugate diameters is equal to the square of the diameter to which the tangent is parallel.

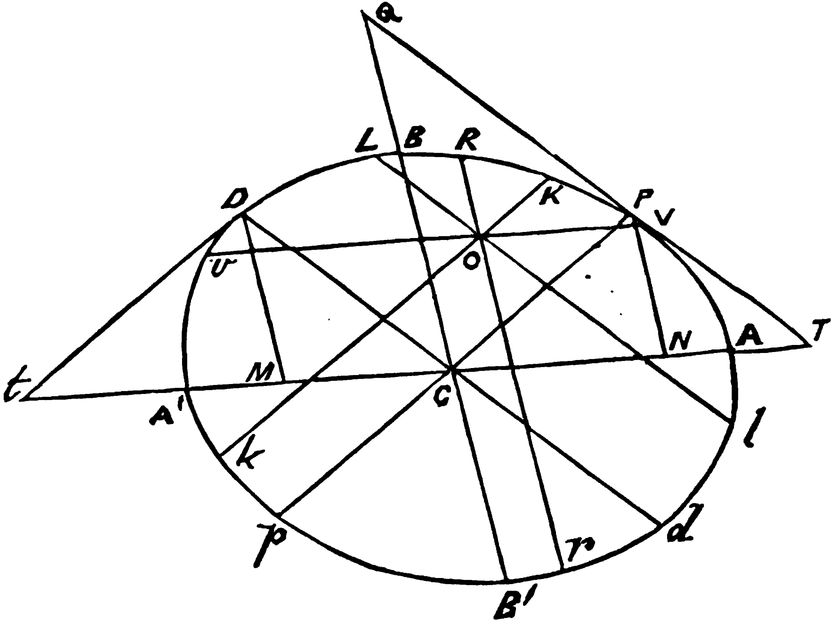
Art. 101.

Fig. 65. From the preceeding results we may also prove the following, viz.:

If through any point O in the plane of the section lines be drawn parallel to any pair of conjugate diameters, the sum of their rectangles remains constant so long as the point O remains fixed.

Draw through O the lines KOk, LOl, ROr, VOv respectively, parto PC, DC, BC, AC.

Fig. 65.



Then by Prop. VII. $KO.Ok : LO.Ol = PC^2 : DC^2$
Therefore $KO.Ok + LO.Ol : PC^2 + DC^2 = LO.Ol : DC^2$
Similarly $RO.Or + VO.Ov : AC^2 + BC^2 = VO.Ov : AC^2$
But by Prop. VII. $LO.Ol : DC^2 = VO.Ov : AC^2$
$\therefore$ Ex æquali $KO.Ok = LO.Ol : PC^2 + DC^2 = RO.Or + VO.Ov : AC^2 + BC^2$
But (by Art. 99) $PC^2 + DC^2 = AC^2 + BC^2$
Therefore $KO.Ok + LO.Ol = RO.Or + VO.Ov$

Hence if BC, AC be a fixed and PC, DC a variable pair of conjugate diameters, $KO.Ok + LO.Ol$ is constant; that is, the sum of the rectangles of the segments of chords through a fixed point and parallel to any pair of conjugate diameters is constant.

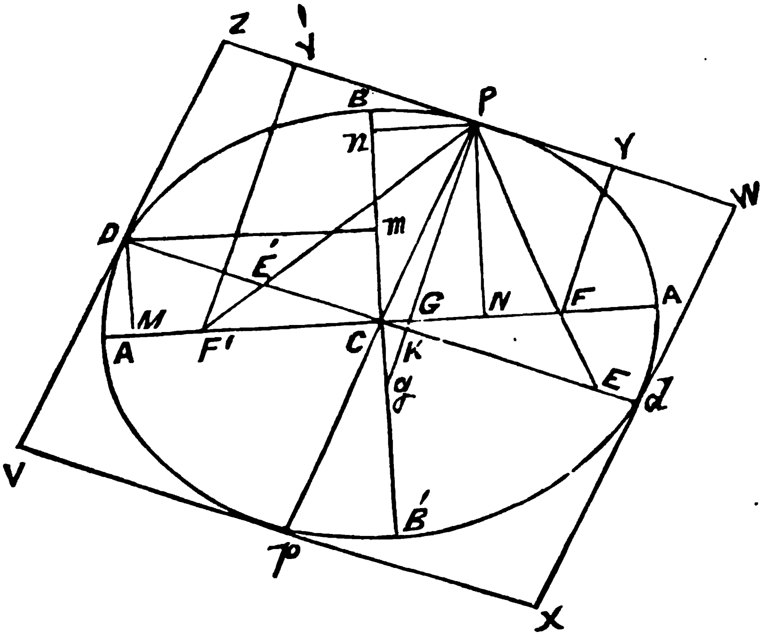
Art. 102.

Fig. 66. Let APA be an Ellipse, and let the normal at P meet the axes in G and g, also let CD be the diameter parallel to the tangent at P and therefore conjugate to Pp, then

$$PG : CD = BC : AC = CD : Pg.$$

Also the parallelogram formed by drawing tangents at the extremities of any pair of conjugate diameters is equal to the rectangle contained by the axes.

Fig. 66.



For since the triangles DCM, PGN are similar

Therefore $PG : CD = PN : CM = BC : AC$ (Art. 98)

Also from the similar triangles Pgn, DCm,

$Pg : CD = Pn : DM = AC : BC$ (Art. 98)

Therefore ex æquali $PG : CD = BC : AC = CD : Pg.$

Fig. 66. Again, let the normal at P meet DCd in K, then by what has just been proved

$PG : BC = CD : AC$

And since (by Art. 96) $PG.PK = BC^2 \therefore PG : BC = BC : PK$

Therefore $CD : AC = BC : PK$

$\therefore$ $CD.PK = AC.BC$

Hence the parallelogram WXVZ is equal to the rectangle AA'.BB'.

Art. 103.

If FP, F'P be joined, then $FP.PF' = CD^2$.

Fig. 66. Let PF, PF' meet CD in E and E', then as has been shown, PE, PE' are each equal to AC, and the triangles PFY, PEK, PF'Y', PE'K are all similar.

Therefore $PF : FY = PE : PK$

$PF' : F'Y' = PE' : PK$

$\therefore$ $PF.PF' : FY.F'Y' = PE^2\ (= AC^2) : PK^2$

$= CD^2 : CB^2$ (Art. 102)

But FY.F'Y' has been shown to be equal to BC^2. (Art. 89)

Therefore $PF.PF' = CD^2$.

EXAMPLES ON THE ELLIPSE.

No.

1.—If PCp be a diameter and the focal distance pF produced meet the tangent at P in T, then $FP = FT$.

2.—In what position of P is the angle FPF' the greatest ?

3—Given the length of the axis, one focus, and a point on the curve, find the centre geometrically.

4.—The sum of the squares of normals at the extremities of conjugate diameters is constant.

5.—If the tangent at B the vertex of the axis minor meet the Latus Rectum produced in Q, then CQX is a right angle.

6.—If a straight line perpendicular to AA' meet AP, $A'P$ produced if necessary in R and S', then RS is bisected by the tangent at P.

7.—Given two conjugate diameters, determine the direction of the axes

8.—Tangents are drawn to con-focal Ellipses from a given point in the axis ; the normals at the points of contact pass through a fixed point.

9.—Having given the foci, describe an Ellipse touching a given straight line.

10.—If CP be a semi-diameter, and AQO be drawn parallel to CP meeting the curve in Q and CB produced in O, then $2CP^2 = AO.AQ$.

11.—If PQ, SR, which are not parallel, make equal angles with either axis, the lines PR, QS, and also PS, QR, will make equal angles with either axis.

12.—If TP, TQ are tangents to an Ellipse, and PQ meets the directrices in Z, Z', then $ZP.Z'P : ZQ.Z'Q = TP^2 : TQ^2$.

13.—If TQ, TQ' are two tangents at right angles, and CT meets QQ' in Z, prove that $ZT = QZ$, and shew that the locus of T is a circle.

14.—Draw a tangent to an Ellipse parallel to a given line.

15.—If the normal at P meet the axis in G, then $PG^2 = FP.PF' - FG'.GF'$

16.—If PN be an ordinate to the axis, and NQ be drawn parallel to AB joined, meeting CB in Q, then $NP^2 = CB^2 - CQ^2 = BQ.QB'$

17.—If the tangent at P meet the axis in T, and YY' are the feet of perpendiculars on the tangent from the foci, then $YT : Y'T = YP : Y'P$.
Prove also that $YP.PY' = FP.PF' - CB^2$

18.—If the tangent at P meet the perpendiculars at A and A', to the axis, in Z, Z', and CB in K, then $AZ : PN = CK : A'Z'$; hence prove that $AZ.A'Z' = BC^2$

19.—If from G the foot of the normal, GK, GK′ be drawn perpendicular to PF, PF′, then PG = PG′ = $\frac{1}{2}$ the Latus Rectum.

20.—The distances of any point on an Ellipse from one of the foci is equal to the ordinate at the same point produced to meet the tangent drawn through the extremity of the focal ordinate.

21.—If the tangent at P the extremity of any diameter CP meet the axis in T, and the tangent at A meet CP produced in Z, then the triangles CAZ, CPT are equal.

22.—Two conjugate diameters of an Ellipse are cut by the tangent at any point P in M, N, prove that the area of the triangle CPM varies inversely as that of the triangle CPN.

23.—The tangent at the vertex meets any two conjugate diameters in Z, Z′, prove that AZ.AZ, = CB^2.

24.—Find the conjugate diameters of an Ellipse that are equal.

25.—Normals are drawn at the extremities of a chord of an Ellipse which is parallel to one of the equi-conjugate diameters, find the locus of their intersection, *i.e.*, of the normals.

26.—If the normal at P meet the axis major in G, prove that PG is an harmonic mean between the perpendiculars on the tangent at P from the foci.

27.—If tangents be drawn at the extremities of a focal chord of an Ellipse, prove that they contain an angle equal to half the supplement of the angle that the chord of contact subtends at the other focus.

28.—The straight lines joining each focus to the foot of the perpendicular from the other focus on the tangent at any point meet on the normal at the point and bisect it.

29.—The sub-normal at any point P is a third proportional to the intercept of the tangent at P on the major axis and half the minor axis.

30.—If tangents be drawn from any point to an Ellipse touching it in Z, Z′, and FZ, F′Z, FZ′, F′Z′ be joined, shew that a circle may be described to touch these four lines, and find its centre.

31.—Equal and similar Ellipses having the same centre intersect at the extremities of diameters at right angles to one another.

32.—If AA′ be the transverse axis, MV a diameter, MN, QL ordinates to AA′ and MV respectively, AE, MT tangents, and QIZ perpendicular to AA′ and meeting AA′ and MV in I and Z, then if QL meet AA′ in R, the triangle QIR = trapezoid IAEZ; E being the point where the tangent AE meets MV produced.

33.—If TP, TQ touch an Ellipse in P and Q, and PQ be joined, and any line be drawn parallel to TQ cutting TP, PQ in G and H and the curve in R and L, then GR, GH, GL are continually proportional.

34.—If two tangents HK, HE touching an Ellipse in D and R, cut any diameter AB produced in K and E, so that AE = BK; and if TG be a third tangent, then the rectangle ET.GK = KH.RE a given quantity.

35.—Any line drawn through the intersection of a pair of tangents to cut an Ellipse is divided harmonically by the curve, the chord of contact and point of concourse of the tangents.

36.—If TP, TQ be two tangents at P and Q, and PQ be drawn, and any line LD cut PQ and the curve in S, A, B, then $SL^2 : SD^2 = AL.LB : BD.DA$.

37.—If VH.VI be two tangents at H, I, and IHD be drawn through the points of contact, and any line VPOL be drawn through V meeting the curve in P and L, the tangents at P and L will meet on ID.

38.—If a trapezium be circumscribed about an Ellipse as ABFE, and if the diagonals AF, BE be bisected, the line drawn through the points of bisection will pass through the centre C.

39.—The external angle between any two tangents to an Ellipse is equal to half the sum of the angles which the chord of contact subtends at the foci.

40.—If the diameter conjugate to CP meet FP and F′P, or these produced in K and K′, prove that FK = F′K′; also show that the circles circumscribing the triangles FCK, F′CK′ are equal.

41.—A Conic is drawn touching an Ellipse at the extremities A, B of the axes and passing through the centre C of the Ellipse, prove that the tangent at C is parallel to AB.

42.—If CP, CD be conjugate diameters, prove that $(FP - AC)^2 + (FD - AC)^2 = FC^2$

43.—If the tangents at three points P, Q, R intersect in K, L, M, prove that
$$PK.MQ.LR = PL.KQ.MR.$$

44.—Find the locus of the centre of a circle that touches FP, F′P produced, and the axis of an Ellipse.

45.—PG is a normal to an Ellipse meeting the axis in G, if a circle on PG cut FP, F′P in K and L respectively, show that KL is bisected by PG and is perpendicular to it.

46.—If P, P′ be two points on an Ellipse, and FP, F′P, FP′, F′P′ be joined, then a circle may be described to touch these four lines.

47.—If FY, F′Y′ be the perpendiculars from the foci on the tangent at P, and PN be the ordinate, show that $PY : PY' = NY : NY'$.

48.—The locus of the centres of all circles, two of whose common chords with a given Ellipse pass through a fixed point, is a straight line through the point.

49.—The locus of the centres of all focal chords in an Ellipse is a similar Ellipse.

50.—The locus of the centre of a circle inscribed in the triangle FPF′ is an Ellipse.

51.—If two parallel tangents be cut by any third tangent to the Ellipse, the rectangle of the segments of the parallel tangents between their points of contact and the third tangent shall be equal to the square of the semi-diameter to which they are parallel; also the rectangle of the segments of the third tangent between its point of contact and the parallel tangents shall be equal to the square of the semi-diameter to which it is parallel.

52.—If a tangent be drawn at D the extremity of the Latus Rectum, meeting the tangents at AA′, the extremities of the axis in H and G, prove that AH = HD and A′G = DG.

53.—If any tangent to an Ellipse meet the tangents at A, A′, the extremities of its axis, in G and H, prove that the circle on GH as diameter passes through the foci.

54.—If CK be the perpendicular on the tangent at P from the centre, and PN perpendicular to the tangent meeting the axis in N, prove that CK.PN = square of semi-axis minor.

55.—The perpendicular on the tangent from the focus is to the distance of the focus from the point of contact as the semi-axis minor to the semi-diameter parallel to the tangent.

56.—If from a point D in the minor axis of an Ellipse, a line DG be drawn to a point G in the transverse axis, such that DG is equal to the sum or difference of the semi-axis of the Ellipse; then if in DG or DG produced, (according as DG is equal to the sum or difference) a point P be taken, such that GP is equal to the difference of the axes, then P is a point on the Ellipse; hence show how to describe the curve mechanically.

57.—If a quadrilateral circumscribe an Ellipse, the angles subtended by opposite sides at one of the foci are together equal to two right angles.

58.—If from any point I in the axis major of an Ellipse, a line be drawn cutting the curve in E and H, and the ordinates ED, HG be drawn; then if K be the middle of DG, and C the centre of the axis, CK : CI = AD.AG : AI^2; and show what this property becomes when the points E and H coincide and IEH becomes a tangent.

59.—The straight line joining the feet of perpendiculars from one of the foci on two tangents is at right angles to the line joining the intersection of the tangents with the other focus.

60.—PQ is the chord of an Ellipse normal at P, LCL′ the diameter bisecting it, show that PQ bisects the angle LPL′, and that LP + PL′ is constant.

61.—If perpendiculars be drawn from a focus upon a pair of conjugate diameters, and produced backwards they will meet on the directrix.

62.—FQ, F′Q are perpendiculars to a pair of conjugate diameters, shew that the locus of Q is an Ellipse, and find its centre.

63.—If AB be a diameter, and if through any point P, two lines APH, BPN be drawn, and also an ordinate DK to AB intersecting the lines AH, BN in E and G, then DE, DK, DG, will be in continued proportion.

64.—If F be the focus, C the centre, QZ any diameter, MN parallel to it through the focus, then AA′, (the axis) QZ, MN are in continual proportion.

65.—Prove that the distance between the two points on the circumference of an Ellipse, at which a given chord not passing through the centre subtends the greatest and least angles, is equal to the diameter which bisects that chord.

66.—Draw common tangents to two Ellipses in the same plane whose axes are equal, each to each, and centres coincident, but whose major axes are inclined to one another.

67.—If PG the normal at P, cut the major axis in G, and if DN′, PN be the ordinates of D and P, CD being conjugate to CP, prove that the triangles PGN, DN′C are similar; and prove that PG has a constant ratio to CD.

68.—If the normal at P in an Ellipse meet the axis minor in g, and if the tangent at P meet the tangent at the vertex A in V, show that $Fg : FC = PV : VA$; F being the focus.

69.—If the distance between the foci of an Ellipse be greater than the axis minor, find the number of positions of the tangent, for which the area of the triangle included between it and the lines drawn from the centre to the feet of the focal perpendiculars on the tangent will be the greatest possible.

70.—The rectangle contained by the radii of the inscribed and circumscribing circles of the triangle FPF′, varies as the square of the diameter conjugate to CP.

71.—The locus of the extremities of central radii of an Ellipse produced in a constant ratio is an Ellipse.

72.—Find the foci of two Ellipses having the same major axes, and that can be inscribed in the same parallelogram.

73.—About the triangle PQR an Ellipse is described, having its centre at the point where the lines drawn from P, Q, R to the middle points of the opposite sides meet; CP, CQ, CR, are produced to meet the Ellipse in K, L, M, prove that the tangents at K, L, M, form a triangle similar to PQR but four times as large.

74.—The acute angles which FP, FQ make with the tangents are complementary; and BC^2 is a mean proportional between the areas of the triangles FPF′, FQF′; and find under what limits the problem is possible.

75.—In an Ellipse, if PQ be drawn at right angles to FP and meet the diameter conjugate to CP in Z, show that PZ varies inversely as the perpendicular on the major axis.

76.—The tangent and normal of a point P of an Ellipse meet the axis in T and G respectively, and show that $CG.CT = CF^2$.

77.—If a tangent at a point P meet the axis in T, and the perpendiculars from the focus and centre meet the tangent in Y and Z, then $TY^2 : PY^2 = TZ : PZ$.

78.—If a circle be drawn through the foci of two confocal Ellipses, cutting the Ellipses in Y, Z, the tangents to the Ellipses at Y and Z intersect on the circumference of the circle.

79.—Given a focus F and three points P, Q, R of an Ellipse, to describe the curve.

80.—If A be the extremity of the major axis, and P any point on an Ellipse, prove that the bisectors of the angles PFA, PF′A, meet on the tangent at P.

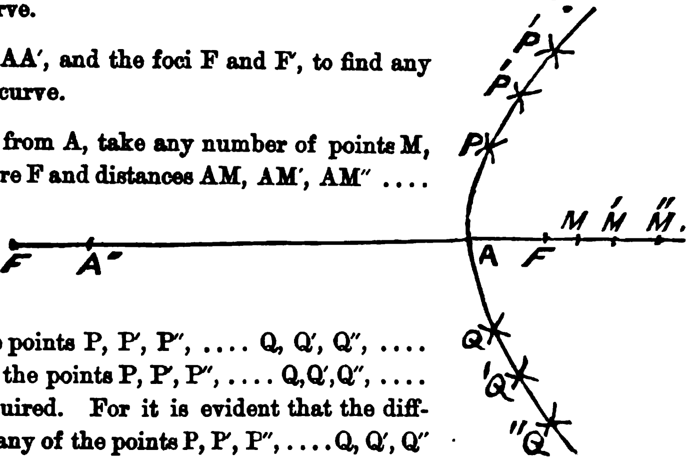
81.—Show that the locus of the intecsection of tangents to an Ellipse that intersect at right angles, is a circle concentric with the Ellipse, and the square of whose radius $= CA^2 + CB^2$.

THE HYPERBOLA.

Art. 104. Construction of the curve.

Fig. 67. Having given the axis AA′, and the foci F and F′, to find any number of points on the curve.

Fig. 67.

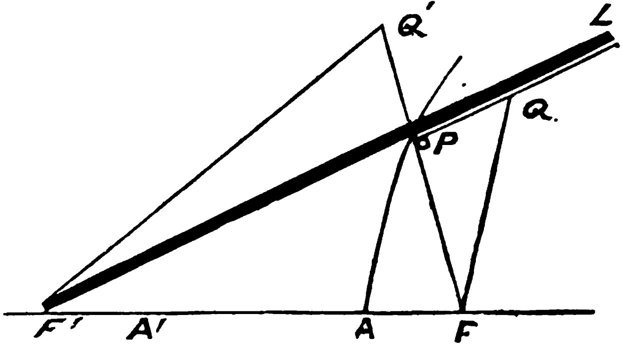


On the other side of F from A, take any number of points M, M′, M″ .... and with centre F and distances AM, AM′, AM″ .... describe circles; also, with centre F′ and distances A′M, A′M′, A′M″, ... describe circles cutting the former in the points P, P′, P″, ... Q, Q′, Q″, ... the curve drawn through the points P, P′, P″, ... Q, Q′, Q″, ... will be the Hyperbola required. For it is evident that the difference of the distances of any of the points P, P′, P″, ... Q, Q′, Q″ .... from F and F′ is equal to AA′ the given axis, which is a property of the Hyperbola demonstrated in Prop. V.

The Hyperbola may be constructed mechanically thus :—

Fig. 68. Let the straight rule F′L be moveable about the fixed point F′, and let AA′ be the axis, and F and F′ the foci of the Hyperbola it is required to describe. At the extremity L of the rule, fasten one end of a piece of thread shorter than the rule by the length of AA′, and fasten the other end of the thread at F, then keeping the string stretched and pressed against the rule by a pencil as at P, whilst the rule moves or revolves about the fixed point F′, the required Hyperbola will be described. For it is evident that in every position of the rule and pencil the difference of the distance F′P and FP always equals AA′.

Fig. 68.



In describing the other branch of the curve, the string must be taken longer than the rod by the length AA′.

Cor.—The difference of the distances of any point from F′ and F is greater or less than the transverse axis according as the point is on the concave or convex side of the Hyperbola.

Let Q be a point on the concave, and Q′ on the convex side of the Hyperbola. (Fig. 68)

Then $F'Q = F'P + PQ$, and $FQ < FP + PQ$ (Eu. i. 20.)

Therefore $F'Q - FQ > F'P - FP$, that is $> AA'$

Again $F'Q' < F'P + PQ'$ (by Eu. i. 20) and $FQ' = FP + PQ'$

Therefore $F'Q' - FQ' < F'P - FP$, that is $< AA'$

Art. 105.

Fig. 69. Let there be an Hyperbola of which AA′ and BB′ are respectively the major and minor axes, F and F′ the foci, XM, X′M′ the directrices, and C the centre; then as already observed, (Art. 26) although the axis minor BC does not meet the curve, still as in the case of the Ellipse, BC^2 is taken equal to $CF^2 - CA^2 = AF.FA'$

And therefore $$CF^2 = CA^2 + CB^2$$

Hence the line CF is equal to the line joining the extremities of the axes, that is to AB or AB′.

Art. 106.

Fig. 69. Also as in the Ellipse, the tangents from the point X, the foot of the directrix, touch the curve at the extremities of the Latus Rectum, or ordinate DFd drawn through the focus; for by Prop. V. the tangents at D and d meet the directrix in the same point as the perpendicular to DFd through the point F does, that is in the point X.

Art. 107.

Fig. 69. The results of Prop. V. for the Hyperbola are as follows :—

(*a*) If PT be a tangent at any point P on the curve, the angle FPT = F′PT.

(*b*) $F'P - FP = AA'$, P being any point on the curve.

(*c*) FT perpendicular to the focal chord PFp meets the directrix in the same point as the tangents at P and p.

(*d*) If PM be a perpendicular on the directrix
$$\begin{aligned} PF : PM = AA' : XX' = AC : CX &= AF : AX \\ &= AF' : A'X. \end{aligned}$$

Art. 108.

Fig. 69. Draw the perpendiculars FY, F′Y′ on the tangent, and let FY produced meet F′P in K, then since the triangles FPY, KPY have two angles of the one equal to two angles of the other, and the side PY common, therefore (by Eu. i. 26) FY = YK and PF = PK; but CF = CF′, therefore CY is parallel to F′P, and $CY = \frac{1}{2}F'K = \frac{1}{2}AA' = CA$; similarly CY′ = CA. Therefore if a circle, which will be the auxiliary circle, be described on AA′ as diameter, it will pass through the points Y, Y′, the feet of the perpendiculars from F and F′.

Let the auxilary circle be described, and produce YC to meet F′Y′ in V; then since YY′V is a right angle, YV is a diameter of the circle, and the point V is on the circle. Hence since KYVF′ is a parallelogram and VF′ = KY = FY

Therefore $$FY.F'Y' = F'V.F'Y' = AF'.F'A' = AF.FA' = BC^2.$$

Art. 109.

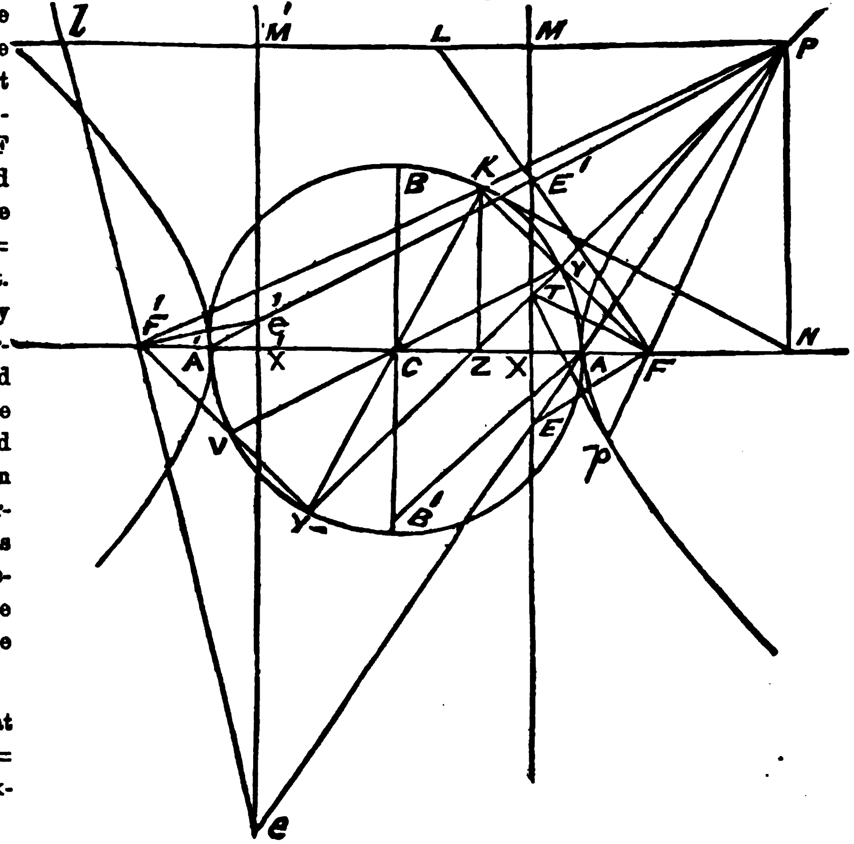
Fig. 69. Again, let PT meet the axis in Z, then since the angle FPZ = F′PZ, we have
$$\begin{aligned} FZ : F'Z &= FP : F'P \\ &= PM : PM' \quad \text{(by } d\text{, Art. 107)} \\ &= XN : X'N \end{aligned}$$

Therefore $$FZ + F'Z : F'Z - FZ = XN + X'N : X'N - XN$$

Or $$2CF : 2CZ = 2CN : 2CX$$

Therefore $$CZ.CN = CX.CF = CA^2$$

Fig. 69.



This may also be proved as in the Ellipse from the fact that AA' is harmonically divided in F and X and bisected in C, and therefore $CX.CF = CA^2 = CZ.CN$. (See Art. 15.) Hence if any number of Hyperbolas be described having the same transverse axis, and an ordinate be drawn cutting the Hyperbolas, the tangents at the points of section will all meet the axis in the same point.

It is also evident that $CN.NZ = KN^2 = AN.NA$ by the auxiliary circle.

Art. 110.

By similar triangles PNZ, FYZ,

$$PN : FY = ZN : ZY$$

Similarly $PN : FY' = ZN : ZY'$

∴ $PN^2 : FY.FY' = ZN^2 : ZY.ZY'$

Or $PN^2 : BC^2 = ZN^2 : KZ^2 \ (= ZY.ZY')$

$= KN^2 : CK^2$ by sim. tri.

$= AN.NA' : CA^2$

Therefore $PN^2 : AN.NA' = BC^2 : CA^2$

Again since $PN^2 : CN^2 - CA^2 = BC^2 : CA^2$

Alt^{do} and comp^{do} $PN^2 + BC^2 : BC^2 = CN^2 : CA^2$

Alt^{do} $PN^2 + BC^2 : CN^2 = BC^2 : CA^2$.

Art. 111.

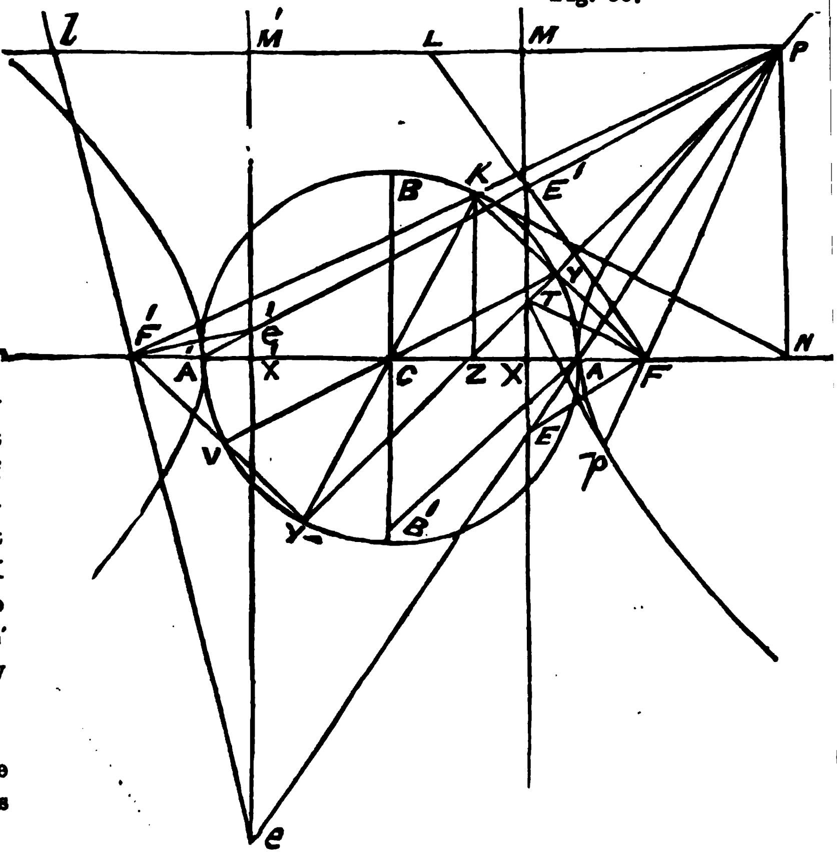
Let P be any point on the curve, and join AP, AP' meeting the directrices in E, E', and e, e', also join FE, FE', F'e, F'e', then $EX.XE' = FX^2$ and $eX'X'e' = F'X'^2$.

For produce FE' to meet RM in L, then

By sim. tri. $FA' : A'X = PL : PM$

But $FA' : A'X = FP : PM$ (by *d*, Art. 107)

Fig. 69.



Therefore $FP = PL$, hence the angle $LFP = FLP = LFA$, that is, FE' bisects the angle PFA; similarly if FE be produced to meet PM, (produced) FE may be shown to bisect the supplement of the angle PFA, therefore EFE' is a right angle; and since FX is perpendicular to EE', therefore $EXXE' = FX^2$; similarly $eX.X'e' = F'X'^2$.

If now PN be the ordinate to the axis at the point P

By sim. tri. $PN : AN = EX : AX$

" $PN : A'N = E'X : A'X$

Therefore $PN^2 : AN.NA' = FX^2 (= EX.XE') : AX.XA'$

But (by *d*, Art. 107) $AF : AX = AC : CX$

Comp^{do} et alt^{do} $AF : AC = FX : A'X$ (1)

By Cor. 2, Art. 15, or Art. 109 $AC : CX = FC : AC$

$\therefore$ (by *d*, Art. 107) $CF : AC = FA : AX$

Comp^{do} $A'F : AC = FX : AX$ (2)

$\therefore$ by (1 and 2) $AF.FA' : AC^2 = FX^2 : AX.XA'$

Therefore by what has been proved above

$$PN^2 : AN.NA' = AF.FA' : AC^2$$

If $BC^2 = AFFA'$ this becomes $PN^2 : AN.NA' = BC^2 : AC^2$

Again since $CF^2 = CA^2 + CB^2$ and $CA^2 = CF.CX \therefore CB^2 = CF.FX$

Therefore $PN^2 : AN.NA' = CB^2 : CA^2$

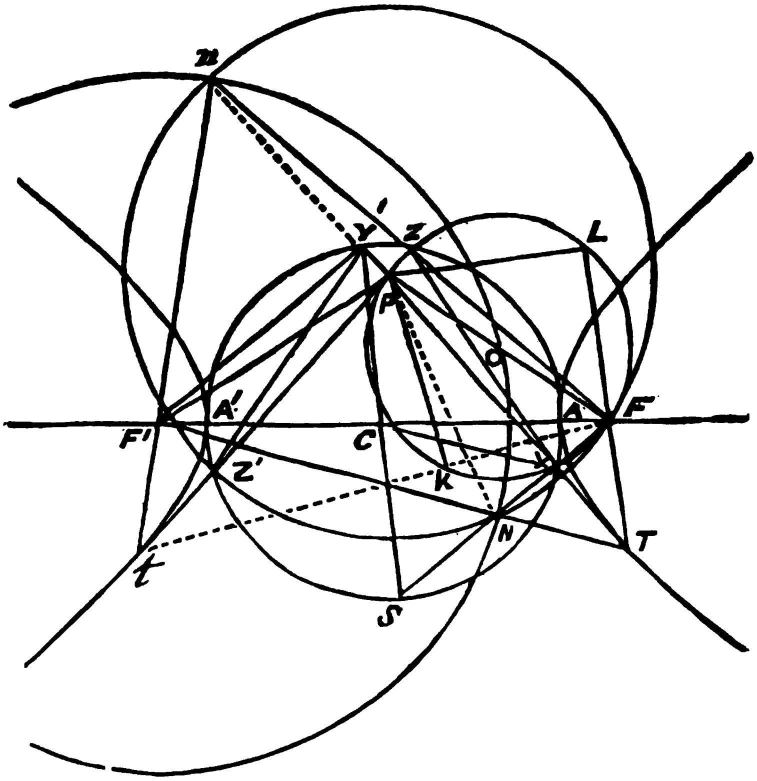
$= FX : CX.$

DEF.—The director circle of an Hyperbola is the circle described from one of the foci, with radius equal to the transverse axis.

Art. 112.

PROB.—To draw tangents to an Hyperbola from a given point P without it.

Fig. 70.



Let AA' be the axis, F and F' the foci, and C the centre of the Hyperbola.

First Method.—With centre F' and radius AA', describe the director circle, and with centre P and radius PF, describe a circle cutting the director circle in the points N and n. Join F'N, F'n meeting the Hyperbola in the points T and t; PT, P^t joined are the tangents required.

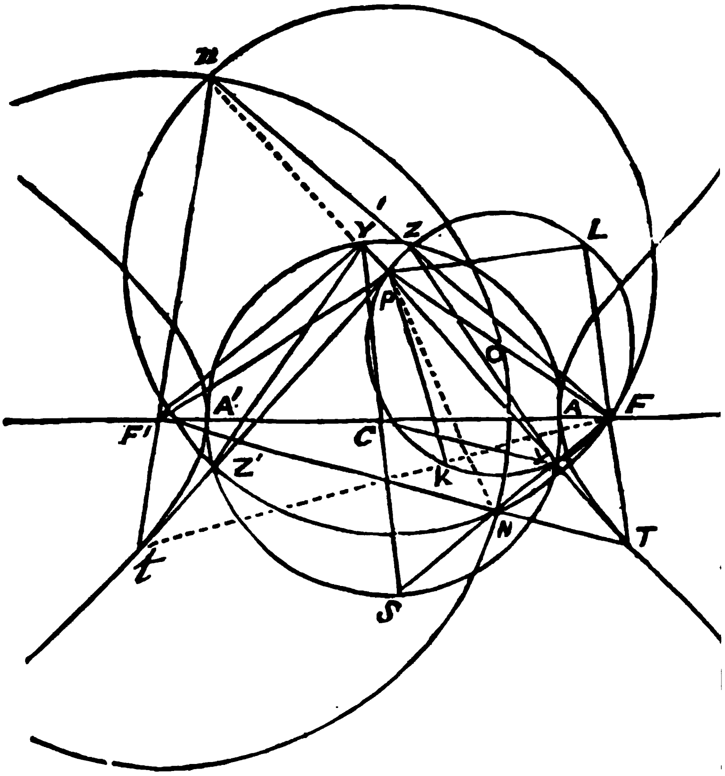
For since F'N = AA' = F'T—FT, therefore FT = TN, and PF = PN; therefore the triangles PFT, PNT, have their three sides equal, and therefore PT makes equal angles with the focal distances of P, and is therefore a tangent at P, (by Art. 107) similarly Pt is a tangent at t.

Second Method.—With centre C and radius CA, describe the auxiliary circle. Join PF and bisect it in O; and with radius PO, describe a circle cutting the auxiliary circle in the points Y and Z; PY, PZ joined and produced are the tangents required.

Join FY and produce it to N, so that YN = FY, and join F'N meeting PY in T, also join YC, F'T.

Then since FY = YN, and YT is common, FY, YT = NY, YT, and the angle FYT = NYT being right angles (by Eu. iii. 31), therefore (by Eu. i. 4) FT = TN, and the angle FTY = NTY = PTF'; but since FC = CF', and NY = YF, therefore F'N is parallel to and equal to 2CY, that is to AA'; therefore F'T—FT = F'N = AA'; therefore the point T is on the Hyperbola, and PT is a tangent since it has been shown that PT makes equal angles with the focal distances of P; similarly PZ (produced) may be shown to be a tangent at t.

Fig. 70.



These methods for drawing tangents to an Hyperbola may be demonstrated also as follows :—

With centre C and radius CA, describe the auxiliary circle.

Join P with one of foci F, and bisect PF in O; with centre O and radius OF, describe a circle cutting the auxiliary circle in Y and Z; also with centre at the other focus F and radius AA′ describe the director circle, and with centre P and radius PF, describe a circle cutting the director circle in two points.

Join FY, FZ and produce them to meet the circle described from P, with radius PF in N and n. Join F′N, F′n meeting PY, PZ in T and t, then the points T and t are on the curve, and PT, Pt are the tangents required.

Join FT, Ft, CY, CZ, then since PY is perpendicular to FN, (for the angle PYF is in a semi-circle) therefore (by Eu. iii. 3) FY = YN; therefore FY, YT = NY, NT, and they contain equal angles; therefore (by Eu. i. 4) FT = TN, and the angle FTY = NTY; wherefore F′T—FT = F′N. But since FY = YN and FC = CF′ therefore F′N = 2CY = AA′, therefore F′T—FT = AA′; hence the point T is on the curve, and the angles FTY, F′TY have been shown to be equal; therefore PT is a tangent at T, and similarly it may be shown that Pt is a tangent at t.

In the course of the proof it has been shown that FN = AA′; therefore the point N, and also n, are on the director circle, being the intersections of that circle with the circle described from P with radius PF.

Therefore either the intersections of the circle O with the auxiliary circle, or the intersections of the circle P with the director circle, are sufficient practically to determine the position of the tangents, for in the one case if PY, PZ be joined and produced, they are the tangents, and in the other case if F′N, F′n be joined, these lines intersect the curve in the points of contact.

Cor.—It appears from the preceding demonstration that the perpendiculars from the foci on the tangents meet them on the auxiliary circle, and if Y, Y′, Z, Z′ be the feet of the perpendiculars, and CY, CY′, CZ, CZ′ be joined, they are each equal to AC.

Art. 113.

Let FY, F′Y′, FZ, F′Z′ be the perpendiculars on the tangents from the foci. Join
Fig. 70. Y′C and produce it to meet the auxiliary circle again in S, and join FS; then since FC, CS are equal to F′C, CY′, and they contain equal angles, therefore (by Eu. i. 4) FS = F′Y′, and the angle CFS is equal to the angle CF′Y′, that is to the angle YFA; therefore the points F, Y, S are in one straight line, and therefore (by Eu. iii. 36) YF.FS = AF.FA′ = BC^2 (by Art. 108); therefore FY.F′Y′ = BC^2, and similarly FZ, F′Z′ = BC^2; that is, the rectangles of the perpendiculars from the foci on the tangents are constant and equal to the square of the semi-axis minor.

Art. 114.

Fig. 70. Again, join YZ, Y′Z′, then since FY.F′Y′ = FZ.F′Z′

FY : FZ = F′Z′ : F′Y′

And the angles YFZ, Y′F′Z′ are equal, for each of the figures FYPZ, F′Y′PZ′ can be inscribed in a circle, (Eu. iii. 22) therefore (by Eu. vi. 6) the triangles FZY, F′Y′Z′ are similar, and the angle FZY = F′Y′Z′, that is, FPY = F′PZ′; hence the tangents from any point P make equal angles with the focal distances of the point.

It is also evident that the rectangle YP.PY′ = ZP.PZ′ that the triangles YPZ, Y′PZ′ are similar; also the angle ZFP = Y′F′P, and the angle YFP = Z′F′P.

Art. 115.

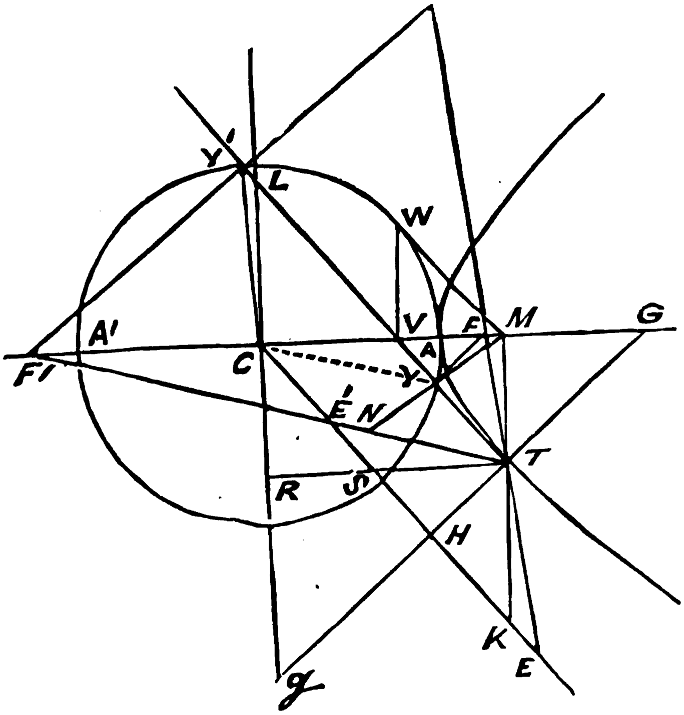
Fig. 70. When the tangents touch the same branch of the Hyperbola, the angles they subtend at the foci are equal as in the Ellipse, (Art. 91) but when the tangents touch opposite branches of the Hyperbola the angles they subtend at the foci are supplementary.

For from the triangles Pnt, PFt which are equal in all respects, the angle Pnt = PFt, and from the equal triangles PnF′, PNF′, the angle PnF′ (Pnt) = PNF′, therefore the angle PFt = PNF′; but from the equal triangles PFT, PNT, the angle PFT = PNT and since PNT, PNF′ are supplementary, therefore also so are the angles PFT, PFt.

That is, the tangents to opposite branches of an Hyperbola subtend supplementary angles at the foci.

Cor.—Hence if FT, Ft or these lines produced meet the circle on PF as diameter in L and K, and PL, PK be joined, each of the angles PLF, PKF being in a semi-circle is a right angle; therefore PL, PK are perpendicular to FT, Ft respectively, but PL = PK, for (the angles PFT, PFt being either equal or supplementary) they subtend equal angles at the point F; that is, if tangents be drawn from any point to an Hyperbola, the perpendiculars from the point on the lines joining the points of contact with one of the foci are equal to one another.

Fig. 71.



Art. 116.

Fig. 71. Let the tangent at any point T of an Hyperbola meet the major and minor axes in V and L respectively.

Draw the ordinate TM and the perpendiculars FY, F'Y' on the tangent, also draw the auxiliary circle which (by Art. 108) will pass through Y and Y', and join YM, YC, FY, F'T, and let the latter meet FY in N; then a circle may be described about the figure FYTM, therefore the angle FMY = FTY = NTY = CYV; therefore the triangles CYM, CVY are similar.

Hence $MC : CY = CY : CV$

Therefore $MC.CV = CY^2 = CA^2$.

Therefore the tangent to the circle from the point M will meet the circle in the same point as the ordinate from V or VW does; and conversely if the ordinate VW of the circle be drawn and the tangent WM meeting the axis in M, then the ordinate of the Hyperbola at M shall meet the curve in the same point as the tangent to it from V does. It is therefore evident how to draw tangents to an Hyperbola from any point V in the axis AA'.

Hence also $CM.MV = MW^2. = AM.MA'$

Art. 117.

Fig. 71. Draw TR parallel to AA' and meeting BC in R, and let the tangent at T meet BC in L.

Then by similar triangles $LC : TM = CV : VM$

Therefore $LC.CR : TM^2 = CM.CV : CM.VM$

Alter^{do} $LC.CR : CA^2 (= CM.CV) = TM^2 : AM.MA'$

$= BC^2 : CA^2$

Therefore $LC.CR = BC^2$.

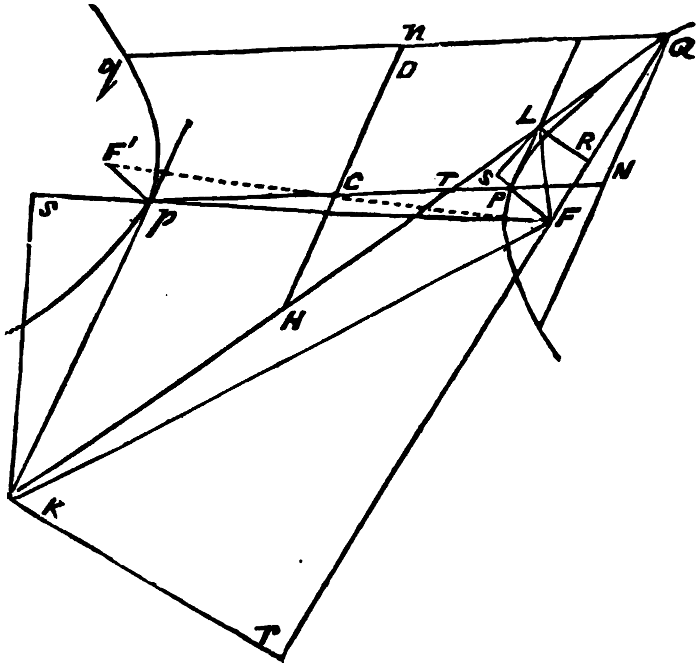


Fig. 72'

Art. 118.

Fig. 72. In like manner it may be shown that if PCp be any diameter of an Hyperbola, and QN an ordinate thereto, if the tangent QT be drawn meeting Pp in T, $TC.CN = CP^2$.

For draw the tangents PL, pK meeting the tangent QT in L and K, then

By Prop. VII. $PL^2 : LQ^2 = pK^2 : KQ^2$
Alternately $PL^2 : pK^2 = LQ^2 : KQ^2$
But by sim. tri. $PL : pK = TL : TK$
Therefore ex æquali $LQ : KQ = TL : TK$ for PL, pK are parallel.
And by parallels $PN : pN = TP : Tp$.

Therefore pP is divided harmonically in T and N, and is bisected in C, therefore $CT.CN = CP^2$.

This may be proved otherwise as follows:—F being a focus. Join FQ, FP, Fp, and draw LR, LS perpendicular to FQ and FP; and Kr, Ks perpendicular to QF and Fp, (produced) and join FK, FL; then (by Art. 115) the angles LFP, LFQ are equal, and the angles at S and R are right, therefore (by Eu. i. 26) $LS = LR$; also (by Art. 115) the angles pFK, KFQ are supplementary, and consequently the angles pFK, KFr are equal, therefore again (by Eu. i. 26) $Kr = Ks$.

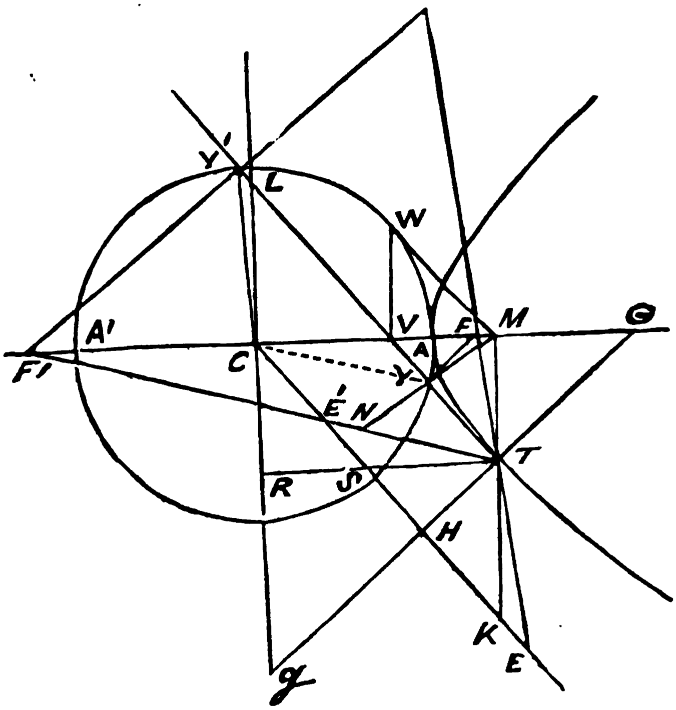
Also since FP is parallel to Fp and PL to pK, the angle FPL is equal to the angle FpK, that is (by *a*, Art. 107) to the angle KpF; therefore the remaining angles LPS, Kps are equal, and the triangles LPS, KPS similar

Therefore by sim. tri. $TL : TK = PL : pK = SL : sK = LR : rK$
$= LQ : KQ$

And the rest follows as in the preceding proof.

NOTE.—If Qnq be drawn parallel to Pp, and Cn parallel to QN and meeting Qq in n and the tangent in H, then by sim. tri. $CH : QN = CT : TN$
∴ $CH.Cn : QN^2 = CT.CN : CN.TN$
By what is proved above $= CP^2 : CN^2 - CP^2$
If therefore CD be taken so that $CD^2 : PC^2 = QN^2 : CN^2 - CP^2$; CD^2 will (by Eu. v. 9) be equal to CH.Cn. It will be shown afterwards that CD is a diameter conjugate to CP and Qq an ordinate thereto, therefore the result of this Art. (118) is true when the diameters meet the ordinates drawn between the two branches of the Hyperbola.

Fig. 71.



Art. 119.

Fig. 71. Through C the centre of the Hyperbola draw ECE' parallel to the tangent VT and meeting F'T, FT (produced) in E' and E; then TE = TE' = AC; also FE = FE'.

Let Y, Y' be the intersections of the tangent with the auxiliary circle, and join CY, CY'; then plainly CE'TY, CETY' are parallelograms, and therefore TE = TE' = CY = AC.

Also since F'T—TT = 2CA = TE + TE', therefore F'T—TE' = FT + TE that is FE' = FE.

Art. 120.

Fig. 71. Draw the normal GTg meeting the axes in G and g and CE in H, also let CE meet TM produced in K.

Then since the figure MHKG may be inscribed in a circle, therefore

$$HT.TG = MT.TK = LC.CR = CB^2$$

Similarly if CE meet TR in S, a circle may be described about SRHg, therefore

$$gT.TH = RT.TS = CM.CV = CA^2.$$

Again since RTg, TMG are similar triangles

$$MG : MC (= TR) = TG : Tg$$
$$= TH.TG : TH.Tg$$
$$= BC^2 : AC^2$$

Also since $\quad MG : MC = BC^2 : AC^2$

Comp^do^ $\quad CG : MC = BC^2 + AC^2 : AC^2$

$$= FC^2 : AC^2$$

Art. 121.

Fig. 71. If a circle be described about the triangle FTF' it will pass through L and g; for the centre is evidently in LC, and it passes through L, since the angle FTL = F'TL, and equal angles stand upon equal circumference; it must therefore also pass through g, since LTg is a right angle.

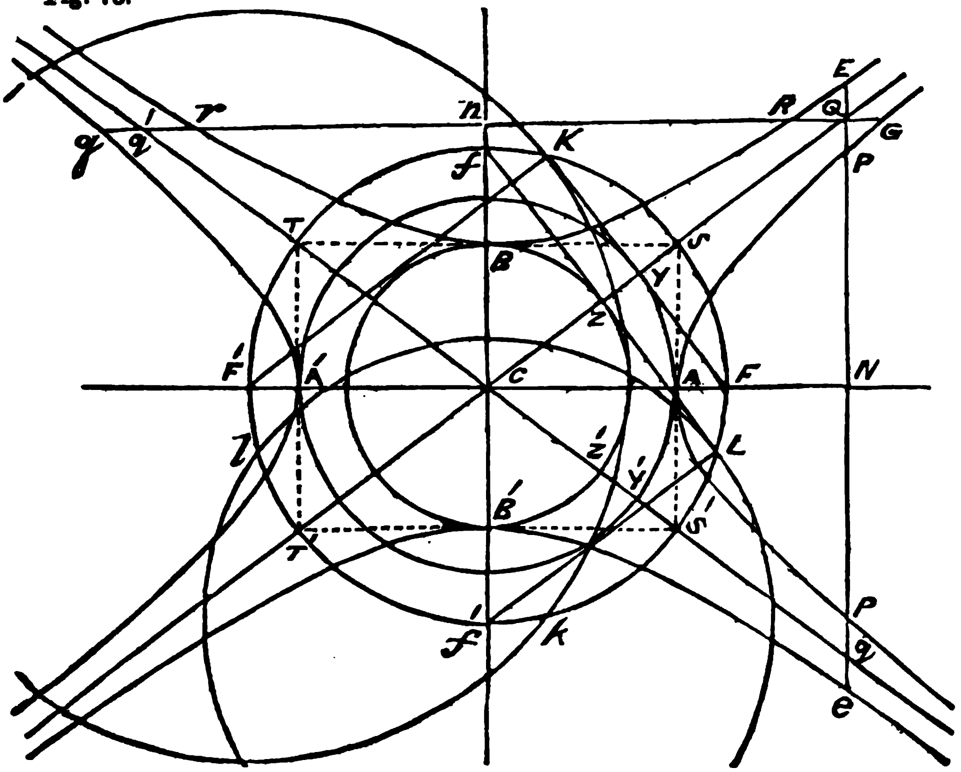
Art. 122.

Since the normal bisects the supplement of the angle FTF', therefore

$$FG : F'G = FT : F'T.$$

CONJUGATE HYPERBOLÆ.

Fig. 73.



Def.—If an Hyperbola be described having for its transverse and conjugate axes respectively, the conjugate and transverse axes of a given Hyperbola, it is called the conjugate Hyperbola.

Art. 123.

If Rn parallel to AA′ be an ordinate of the conjugate Hyperbola

Fig. 73. We have as in (Art. 110) $Rn^2 : Cn^2 - CB^2 = CA^2 : CB^2$

And comp^{do} et alt^{do} $Cn^2 : Rn^2 + AC^2 = BC^2 : CA^2$

Art. 124.

As in the original Hyperbola $CA^2 + CB^2 = CF^2$, so in the conjugate Hyperbola

Fig. 73. $CA^2 + CB^2 = Cf^2$, f and f′ being the foci of the conjugate Hyperbola; therefore the foci of an Hyperbola and its conjugate all lie at the same distance from the centre.

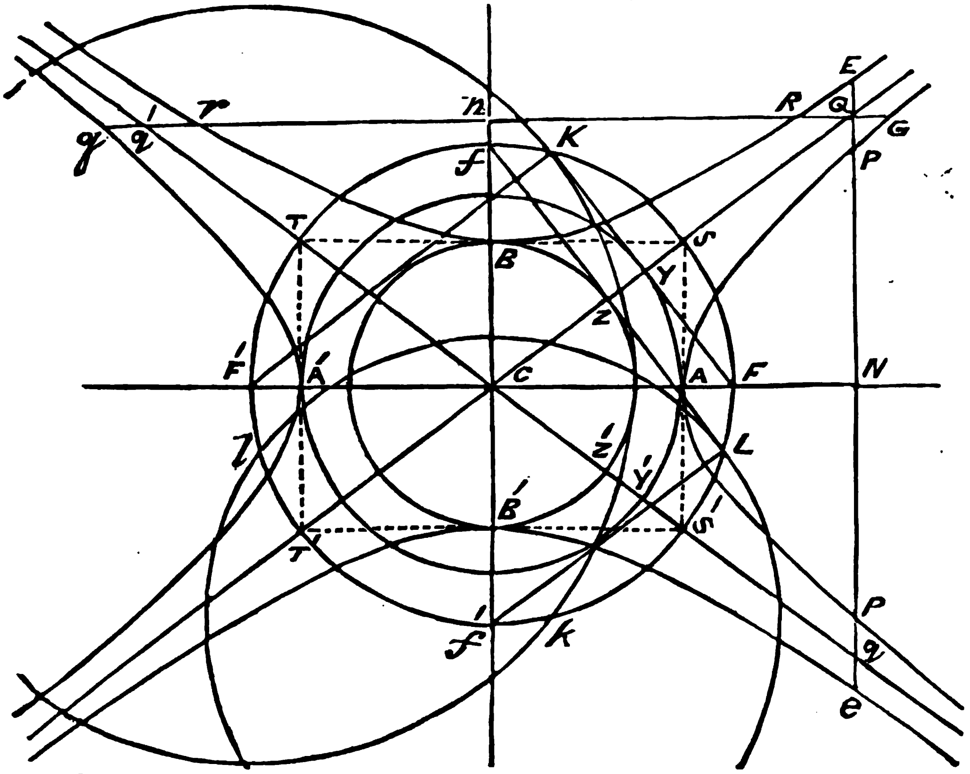
Art. 125.

ASYMPTOTES.—It was shown in (Art. 112) that the points of contact of tangents

Fig. 73. from any point P are obtained by describing a circle about P with radius PF to cut the director circle about F′ in two points, then joining F′ with the intersection of the circles, and that the joining lines (produced) will meet the curve in the points of contact required; or in other words, the lines joining F′ with the intersections of the circles meet the tangents in the points of contact.

Now when the point from which the tangents are to be drawn is the centre C, this construction fails for this reason, that it will be found in this case the lines joining F′ with the intersections of the circles become parallel to the tangents themselves.

Fig. 73.



With centre C and radius CF describe a circle cutting the director circle about F' in K, k, and join F'K, F'k, FK, Fk. Describe the auxiliary circle with centre C and radius CA, and draw the tangents thereto, FY, F'Y' and join CY, CY'.

Then by Art. 112, CY, CY' are the tangents from the point C.

But since $FF' = 2CF$ and $F'K = AA' = 2CA$

Therefore $FF' : F'K = CF : CY$

And the angles FKF', FYC are right angles, therefore (by Eu. vi. 7) the triangles FKF', FYC are similar, and therefore the angle $FCY = FF'K$; that is, CY is parallel to F'K.

In the same manner F'k is parallel to CY'; hence since F'K, CY and Fk, CY' are parallels they do not meet, and consequently CY, CY' do not meet or touch the curve.

It may also be shown that CY, CY' do not meet the conjugate Hyperbola.

For let L and l be in the intersections of the circle, the centre of which is C and radius Cf, with the director circle of which the centre is f' and radius BB'. Join fL, then since $ff'^2 = 4(CA^2 + CB^2)$ and $f'L^2 = 4CB^2$, and the angle fLf' in a semi-circle is right; therefore $fL^2 = 4CA^2$ and $fL = 2CA$; hence $FF' : F'K = ff' : fL$, and the angles at K and L are right; therefore (by Eu. vi. 7) the triangles F'KF, fLf' are similar; therefore fL is parallel to FK, (for they make equal angles with AA') also f'L is parallel to F'K, that is to CY; and since $fC = \frac{1}{2}ff'$, $CZ = \frac{1}{2}f'L = CB$, therefore f'L touches the auxiliary circle on BB'; but Z is the point of intersection of the auxiliary circle on BB' with a circle on fC as diameter; therefore by (Art. 112) CZ will be the tangent from C to the conjugate Hyperbola; and exactly in the same way that it was shown that CY did not meet the Hyperbola the foci of which are F and F, it may also be shown that CZ does not meet the conjugate Hyperbola, and CY, CZ are the same straight line, therefore CY, CY' do not meet either the Hyperbola or its conjugate.

Art. 126. From a point Q in the line CY draw QN, Qn at right angles to the axes CA and CB

Fig. 73. respectively, and meeting the Hyperbola in P and p and its conjugate in R, r.

Then $QP.Pq = BC^2$ and $QR.Qr = AC^2$

For since $BC^2 = CF^2 - CA^2 = FY^2 = SA^2$

Therefore by sim. tri. $QN^2 : BC^2 (= AS^2) = CN^2 : AC^2$

$\therefore \quad NQ^2 - BC^2 : BC^2 = CN^2 - AC^2 : AC^2$

$\qquad = PN^2 : BC^2$

Therefore $QN^2 - BC^2 = PN^2$ and $BC^2 = QN^2 - PN^2 = QP.Qp$

$\qquad = QP.Pq$

Similarly $Qn^2 : AC^2 (= BS^2) = Cn^2 : BC^2$

$Qn^2 - AC^2 : AC^2 = Cn^2 - BC^2 : BC^2$

By Art. 123 $\qquad = Rn^2 : AC^2$

Therefore $Qn^2 - AC^2 = Rn^2$ and $AC^2 = Qn^2 - Rn^2 = QR.Qr = QR.Rq'$

Similarly if Pp and Qq′ be produced to meet the conjugate Hyperbola in E, e, and G, g, it may be shown that $EQ.Qe = BC^2$ and $GQ.Qg = AC^2$.

Also since Qe = qE and Qg = q′G, therefore $QE.Eq = BC^2$ and $QG.Gq = AC^2$; and it will be observed that these results are true when QN falls between C and A, and Qn between C and B.

If the point Q be taken further from the centre, then the lines QN, Qp are increased in length, and by supposing the point Q continually to recede from the centre, the lines QN, Qp will be increased indefinitely or without limit; but the rectangle QP.Pq is constant, being always equal to BC^2; therefore as Q recedes from the centre QP must diminish without limit. and the same is true of QR.

Hence as the curves recede from the centre they approach nearer and nearer to the line CY, and that without limit; therefore the line CY and also CY′ continually approaches the Hyperbola and its conjugate, but never actually meet them. Hence the following—

Def.—The lines CY, CY′ are called the ASYMPTOTES of the Hyperbola and its conjugate.

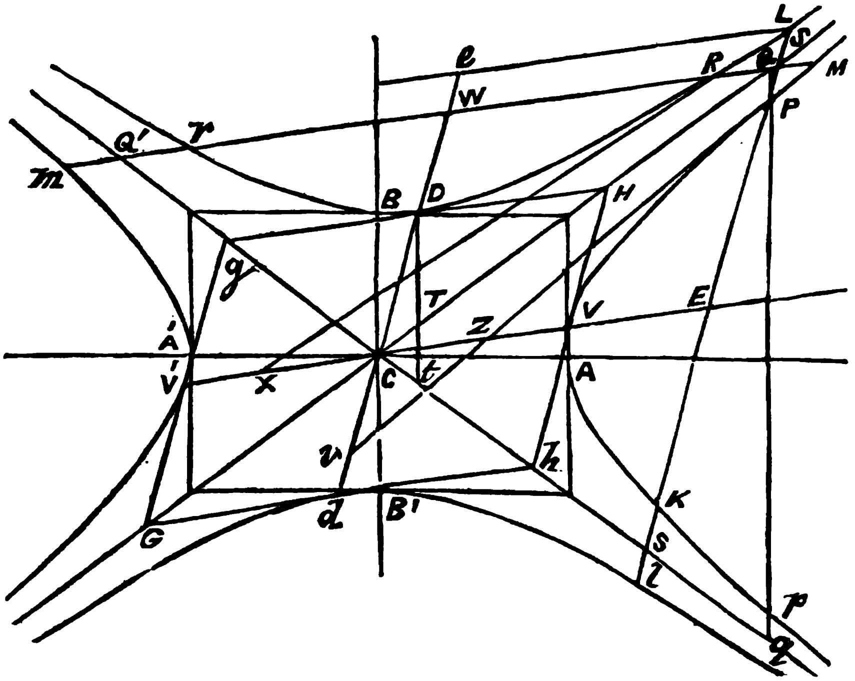
It is clear from what has been above shown that an Asymptote may be considered as a tangent at a point of the curve infinitely distant from the vertex, and conversely the tangent at a point infinitely distant from the vertex passes through the centre.* This will be found to agree with Art. 34, for the points of contact of that Art. are obtained by drawing FY, FY′ parallel to SR, SR′ (fig. 31) to meet SV, SV′ in Y and Y′; but when the tangents are drawn from the centre, the points V and R and V′ and R′ coincide, and are in fact the points Y and Y′ in (fig. 73), (the line YY′ being evidently a directrix) so that in this case FY, FY′ being parallel to SV, SV′ (fig. 31) meet the same and the curve at infinity, that is CY, CY′ (fig. 73) touch the curve at points infinitely distant from the vertex.

It is easy to see that the Asymptotes are the diagonals of the parallelogram SS′TT′ described on the axes of the Hyperbola, and hence also it may be inferred that conjugate Hyperbolæ have the same Asymptotes.

---

* Note.—That this is the case may also be seen from Art. 116, where it is proved that (fig. 71) $MC.CV = CA^2$. Now the less MC becomes by the recession of the point P from the vertex of the curve, the greater will be MC, and when the points V and C coincide, the points M and P will be infinitely distant from the centre and the vertex, and the tangent from C will coincide with the Asymptote, and the same is true of the conjugate Hyperbola.

Fig. 74.



Art. 127.

Fig. 74. Let there be an Hyperbola and its conjugate, and draw any line cutting the Hyperbola in P and K and the Asymptotes in S and s; and let CD be the diameter parallel to SPs; then SP.PS = CD^2, and PS = Ks.

Through P and D draw QPq, DTt perpendicular to the axis AA′ and meeting the Asymptotes in Q, q, T, t.

Then by sim. tri. SP : QP = CD : DT

Ps : Pq = CD : Dt

∴ SP.Ps : QP.Pq = CD^2 : DF.Dt

But QP.Pq = DT.Dt = BC^2, (by Art. 125) therefore SP.Ps = CD^2

Similarly SK.Ks = CD^2, therefore SP.Ps = SK.Ks

But if E be the middle point of Ss, then

$SE^2 - EP^2$ = SP.sP = SK.Ks = $SE^2 - EK^2$

Therefore $EP^2 = EK^2$ and EP = EK

And therefore PS = KS.

If SPs be produced both ways to meet the conjugate Hyperbola in L and l, then it may be be shown in a similar manner that LS.Sl = Ls.sl = CD^2.

Hence LS = ls, and SP, Ks have been proved equal; therefore PL = Kl

And also EL = El.

Art. 128.

Fig. 74. Draw the diameter CE meeting the curve in V, and the tangent HVh meeting the Asymptotes in H and h; then PK being bisected in E is an ordinate to CV, and therefore PK is parallel to the tangent VH. (Prop. VIII.)

Hence since SE = Es ∴ HV = Vh

And since SP.Ps = CD^2 always, therefore $HV^2 = CD^2$

And HV = CD.

Art. 129.

FIG. 74. Draw the ordinate QRr to the diameter CD meeting it in W and the Hyperbola in Mm, the conjugate Hyperbola in Rr, and the Asymptotes in Q, Q′; then since MW = Wm and MR = mr (by Art. 127) ∴ WR = Wr, therefore Rr is an ordinate to CD in the conjugate Hyperbola.

Hence the diameters CV, CD are conjugate in both Hyperbolas, that is, each bisects chords parallel to the other in both curves.

Therefore CV, CD are parallel to the tangents at D and V respectively, if therefore HD be joined, since HV = CD (by Art 128), the figure DCVH is a parallelogram.

Therefore completing the figure HhGg, it is evident that the Asymptotes are the diagonals of the parallelogram formed by tangents at the extremities of any pair of conjugate diameters.

Art. 130.

FIG. 74. If DV, AB be joined, it is manifest that they are parallel to the Asymptote CQ′, and are bisected by the Asymptote CQ.

Art. 131.

FIG. 74. Since $SE^2 : HV^2 (CD^2) = CE^2 : CV^2$

$$SE^2-CD^2 : CD^2 = CE^2-CV^2 : CV^2$$

But (by Art. 127) $SE^2-CD^2 = PE^2$

Therefore $PE^2 : CD^2 = CE^2-CV^2 : CV^2$

And $PE^2 : CE^2-CV^2 = CD^2 : CV^2$.

Compare Arts. 110 and 111, where similar results are obtained for the axes.

Art. 132.

FIG. 74. Let the tangent at P meet the conjugate diameters CV, CD in Z and v, then $PE.Cv = CD^2$.

For $Cv : PE = CZ : ZE$

∴ $Cv.PE : PE^2 = CE.CZ : CE.ZE$

But $CE.CZ = CV^2$ and $CE.EZ = CE^2-CE.CZ = CE^2-CV^2$

Therefore $Cv.PE : PE^2 = CV^2 : CE^2-CV^2$

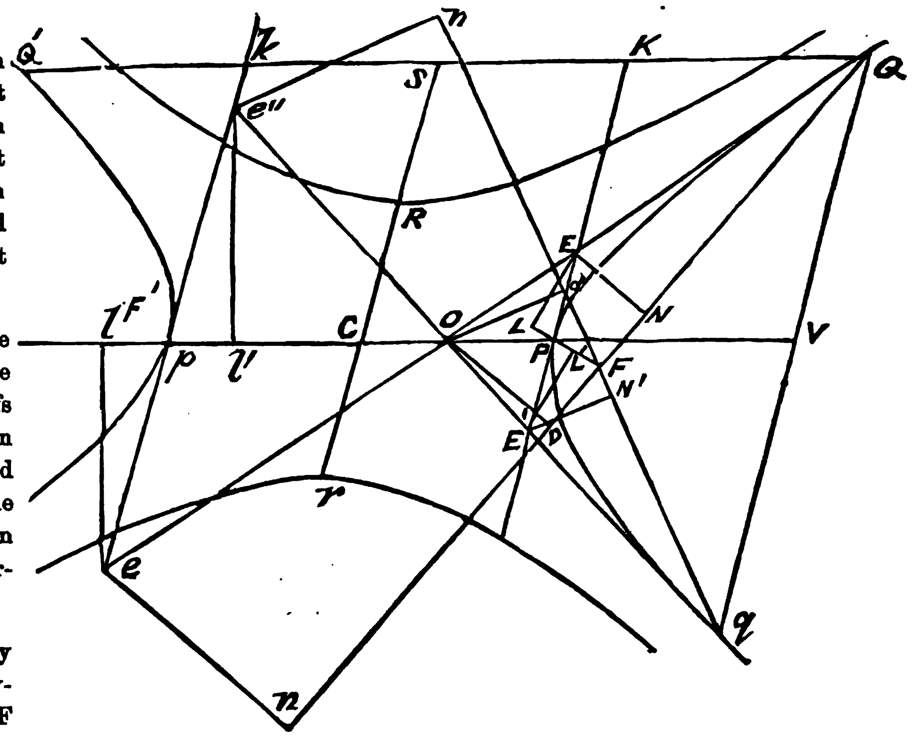
$$= CD^2 : PE^2$$

Therefore $Cv.PE = CD^2$.

In like manner if the tangent at any point L of the conjugate Hyperbola meet VC in X, it may be shown that $CE.CX = Le.CX = CV^2$.

CONJUGATE DIAMETERS.

Art. 133.
Fig. 75.



It has been shown in Prop. VIII. that in any Conic section the diameters bisect all chords drawn in the sections parallel to the tangents at their extremities.

In the cases of the Parabola and Ellipse independent proofs have been given in Arts. 72 and 97, and the following is the analogous proof in case of the Hyperbola.

Let PCp be any diameter of an Hyperbola of which F and F' are the foci and C the centre. Draw the tangents EPE', epe', which (by Art. 23) will be parallel; also any chord QVq parallel to EE' or ee' and meeting Pp (produced) in V, then Qq is bisected in V.

Draw the tangents OQ, Oq to the Hyperbola, and let them meet the parallel tangents EE', ee' in the points E, E', e, e'.

Join QF, Fq, and draw OD, EN, en; Od, E'N', e'n' perpendiculars upon them (produced if necessary); also join FP, Fp,*and let fall the perpendiculars EL, E'L'; el, e'l' upon them.

Then (by Art. 115) $OD = Od, EN = EL, E'N' = E'L', en = el,$ and $e'n' = e'l'$

But by sim. tri.

$$\begin{aligned} OD : EN &= OQ : EQ \\ &= Oq' : E'q \\ &= Od : E'N' \end{aligned}$$

Therefore $EN = E'N'$ and $\therefore$ $EL = E'L'$, and consequently (by Eu. i. 26) $EP = E'P$; in the same way ep may be shown to be equal to e'p.

Therefore Pp must pass through O and therefore bisects Qq, which is parallel to EE' or ee' in the point V.

---

* The line Fp is omitted in the figure.

In the preceding proof the ordinate Qq is taken within or on the concave side of one of the branches of the Hyperbola. If however QQ′ be drawn parallel to PCp and meeting both branches of the Hyperbola in Q and Q′, it can easily be shown that QQ′ is bisected by, and consequently will be an ordinate to, the diameter conjugate to Pp.

For let RCr be the diameter conjugate to Pp, and therefore parallel to the tangents EE′ and ee′, and let RCr meet QQ′ in S, and QQ′ meet EE′ and ee′ in K and k.

Then by Prop. VII. $QK.KQ : PK^2 = Qk.kQ' : pk^2$

But PK = pk, since PKkp is a parallelogram; therefore $QK.KQ' = Qk.kQ'$,

That is $QK.kQ' + QK.Kk = QK.Q'k + Q'k.Kk$

And taking away the common rectangle QK.kQ′, Kk.kQ′ = Kk.KQ; therefore kQ′ = KQ, and SK = Sk, since CP = Cp; therefore QS = Q′S, that is, QQ′ is bisected in the point S; and in the same manner all lines drawn between the two branches of the curve may be shown to be bisected by the diameters conjugate to those to which they are parallel.

Since (by Art. 129) conjugate diameters of an Hyperbola are also conjugate diameters of the conjugate Hyperbola, it will be seen that those diameters (as RCr) of an Hyperbola which do not meet the curve but meet the conjugate Hyperbola have their magnitude determined thereby. (See Arts. 26 and 105.)

And it will be observed generally from Arts. 129 and 133, that if a line as Qq be drawn cutting one of the branches of an Hyperbola and be produced to meet both branches of the conjugate Hyperbola, that this line is an ordinate within one of the branches of the Hyperbola, but between the branches of the conjugate curve to the same diameter CV; and likewise QQ′ in the same manner is an ordinate within the conjugate Hyperbola, but between the branches of the Hyperbola to the same diameter CS.

Art. 134.
Fig. 76. Let VCV′, DCd be one pair of conjugate diameters, and PCp, QCq another pair, and draw PN, QM ordinates to VCV′, then if the tangents at P and Q meet CV in T and t

$$CN^2 = CM.Mt \text{ and } CM^2 = CN.NT$$

Also $PN : CM = DC : CV$

And $QM : CN = DC : CV$

For since (by Art. 118) $CN.CT = VC^2 = CM.Ct$

Therefore
$$\begin{aligned} CM : CN &= CT : Ct \\ &= PT : CQ \quad \text{by sim. tri.} \\ &= PN : QM \quad ,, \\ &= CN : Mt \quad ,, \end{aligned}$$

Therefore $CN^2 = CM.Mt = CM^2 + CM.Ct = CM^2 + VC^2$

And therefore $CM^2 = CN^2 - VC^2 = CN^2 - NC.CT = CN.NT$

But $PN^2 : CN^2 - VC^2 = DC^2 : CV^2$

∴ $PN : CM = DC : CV$

And similarly $QM : CN = DC : CV$

Hence $PN.CN = CM.MQ.$

Art. 135.
Fig. 76. In the above proof it has been shown that $VC^2 = CN^2 - CM^2$; similarly if Pn, Qm be ordinates of DC, it may be shown that $DC^2 = Cm^2 - Cn^2$, that is, $QM^2 - PN^2 = DC^2$.

These relations are true of any pair of conjugate diameters including the axes; hence if VCV′, DCd coincide with the axes AA′, BB′, then since in that case the angles at M and N are right, we have

$$CP^2 = CN^2 + NP^2 \text{ and } CQ^2 = CM^2 + QM^2$$

And by last Art. $CN^2 - CM^2 = AC^2$ and $QM^2 - PN^2 = BC^2$

Therefore $CP^2 - CQ^2 = AC^2 - BC^2$

That is the difference of the squares of any pair of conjugate diameters is constant and equal to the difference of the squares of the axes.

Art. 136.
Fig. 76. Also if through any point O four lines ZOz, ROr, KOk, LOl be drawn parallel to two pairs of conjugate diameters, then

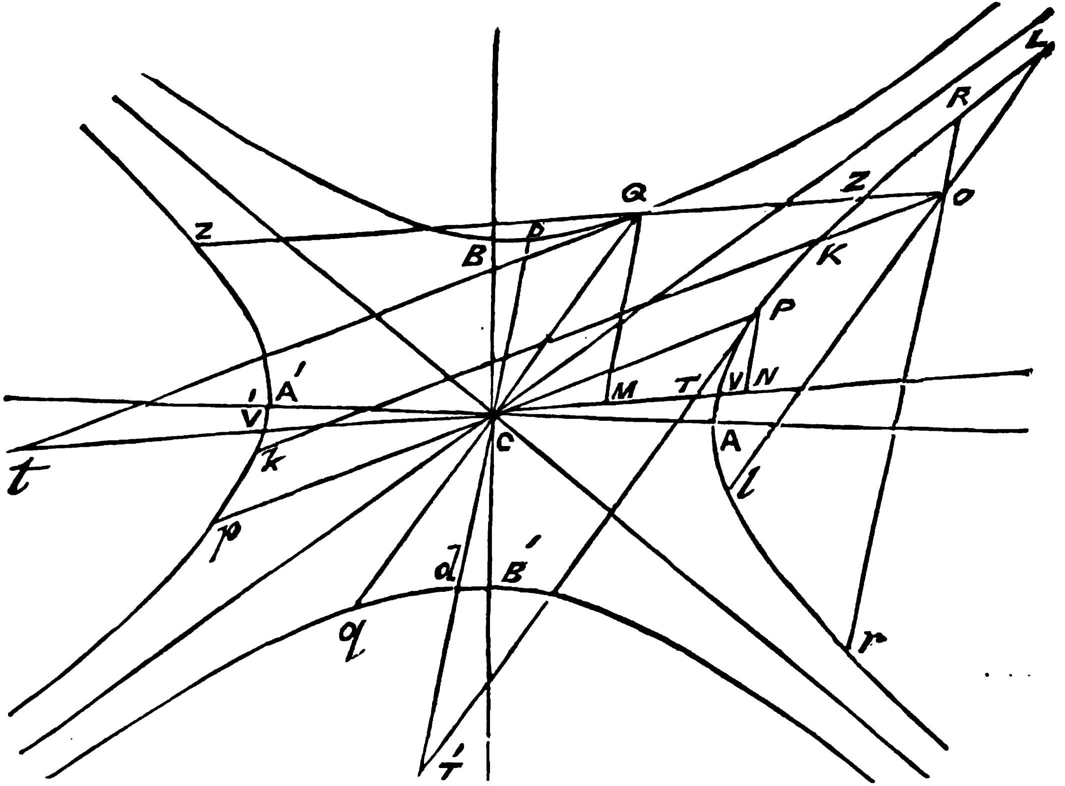
$$ZO.Oz - RO.Or = KO.Ok - LO.Ol$$

For by Prop. VII. $ZO.Oz : RO.Or = CV^2 : DC^2$

∴ div^do^ $ZO.Oz - PO.Or : RO.Or = CV^2 - DC^2 : DC^2$

Similarly $KO.Ok - LO.Ol : LO.Ol = CP^2 - CQ^2 : CQ^2$

Fig. 76.



But by Prop. VII. $RO.Or : DC^2 = LO.Ol : CQ^2$
Therefore ex æquali $RO.Oz—RO.Or : CV^2—DC^2 = KO.Ok—LO.Ol : CP^2—CQ^2$
But $CV^2—DC^2 = CP^2—CQ^2$
Therefore $ZO.Oz—RO.Or = KO.Ok—LO.Ol$

Hence the difference of the rectangles of the segments of a pair of chords drawn through a fixed point parallel to any pair of conjugate diameters is constant.

Art. 137.
Fig. 76. Produce the tangent at P to meet the pair of conjugate diameters CV, CD in TT', and let CQ be the diameter conjugate to CP, then $TP.PT' = CQ^2$.

For by sim. tri. $PT : PN = CQ : QM$
$PT' : CN = CQ : CM$
$\therefore$ $TP.PT' : PN.NC = CQ^2 : QM.MC$
But (by Art. 134) $PN.NC = QM.MC$
Therefore $TP.PT' = CQ^2$.

Art. 138.
Fig. 77. Let the normal at P meet the axes in G and g, and CD be the diameter parallel to the tangent at P, and therefore conjugate to Pp, then

$$PG : CD = BC : AC = CD : Pg$$

Also the parallelogram formed by drawing tangents at the extremities of any pair of conjugate diameters is equal to the rectangle contained by the axes.

And further if PF, PF′ be joined, then $FP.PF' = CD^2$.

Draw the ordinates PN, DM to the axis AA′, and Pn to BC, then by similar triangles PNG, DCM.

$$PG : CD = PN : CM$$

By (Art. 134) $\qquad = BC : AC.$

Again by sim. tri. Pgn, DCM,

$$CD : Pg = DM : Pn$$

By (Art. 134) $\qquad = BC : AC$

Therefore $\qquad IG : CD = BC : AC = CD : Pg$

And therefore $\qquad PG.Pg = CD^2$

Art. 139.
Fig. 77. Let the normal at P meet CD in K, then by the last Art.,

$$PG : BC = CD : AC$$

But (by Art. 120) $\qquad PG.PK = BC^2 \;\therefore\; PG : BC = BC : PK$

Therefore $\qquad CD.PK = BC.AC.$

Hence completing the figure, it appears that the parallelogram ZVXW formed by tangents at the extremities of conjugate diameters is constant and equal to the rectangle of the axes.

Art. 140.
Fig. 77. Let PF, PF′ produced meet CD or it produced in E and E′, and draw FY, F′Y′ perpendiculars from F and F′ on the tangent at P.

Then as has been shown, (Art. 119) $\qquad PE = PE' = AC$

And the triangles PFY, PEK, PF′Y′, PE′K being similar,

$$PF : FY = PE : PK$$

$$PF' : F'Y' = PE' : PK$$

$\because \qquad PF.PF' : FY.F'Y' = PE^2 : PK^2$

$\qquad = AC^2 : PK^2$

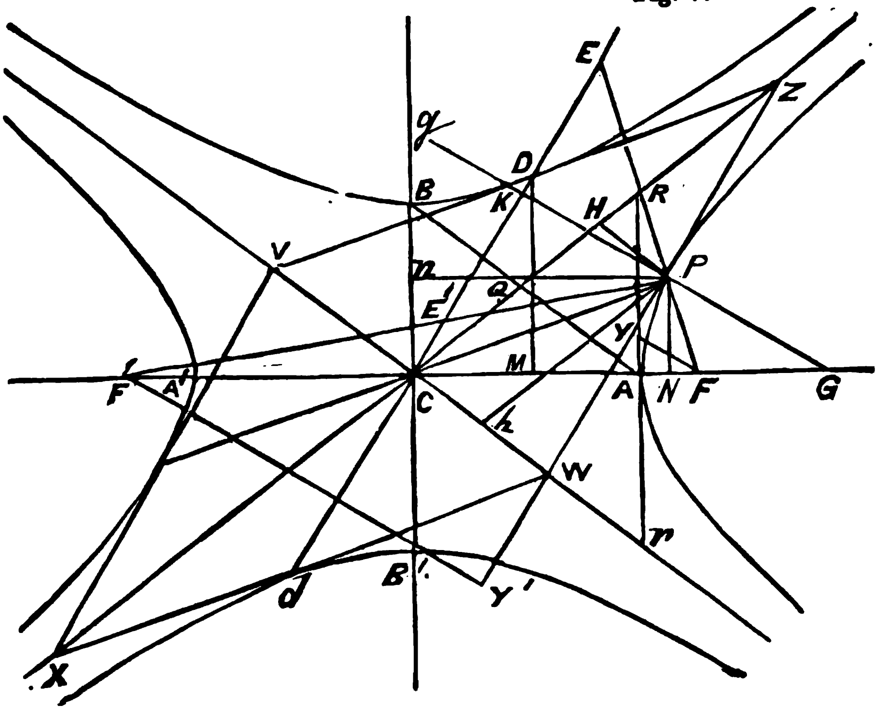
By (Art. 139) $\qquad = CD^2 : BC^2$

But $\qquad FY.F'Y' = BC^2$

Therefore $\qquad PF.PF' = CD^2.$

That is the rectangle of the focal distances of any point on an Hyperbola is equal to the square of the semi-diameter conjugate to that passing through the point.

Fig. 77.



Art. 141.

Let the tangent at P meet the Asymptotes in Z and W, then $ZC.CW = CF^2$.

Draw the tangent RAr meeting the Asymptotes in R and r, then (by Art. 139) the triangles ZCW, RCr are of equal area, being respectively the fourth parts of equal parallelograms, viz., ZVXW, and the rectangle of the axes.

∴ (By Eu. vi. 15) $ZC : CR = CR (= Cr) : CW$
∴ $ZC.CW = CR^2 = CF^2$.

Also if PH.Ph be drawn parallel to the Asymptotes, then since ZW is bisected in P, (Art. 128.)

$CW = 2PH$ and $CZ = 2Ph$
Therefore $CF^2 = ZC.CW = 4PH.Ph$
And hence $PH.Ph = AQ^2 = AQ.QC$
Where Q is the intersection of BA and CZ.

THE EQUILATERAL OR RECTANGULAR HYPERBOLA.

Art. 142.

FIG. 78. If an Hyperbola be described having equal axes, the angle between the Asymptotes is a right angle, and the curve is called equilateral or rectangular.

In this case it is also evident that the conjugate Hyperbola will be equalateral or rectangular since it has the same axes and Asymptotes.

In the the equilateral Hyperbola $CF^2 = 2AC^2$ and $FA^2 = FA^2 = 2AX^2$

For $CF^2 = CA^2 + CB^2 = 2AC^2$

And $FA : AX = FC : CA$

$\therefore$ $FA^2 : AX^2 = 2AC^2 : CA^2$

$\therefore$ $FA^2 = 2AX^2$

Art. 143.

FIG. 78. Describe the auxiliary circle and draw the tangent FD which will touch the circle at the point of its intersection with the directrix, since $FC.CX = CA^2$. (Art. 109)

But $FD^2 = A'F.FA = BC^2 \therefore FD = BC = CD$

Therefore the triangle CDF is isosceles and therefore each of the angles DCF, DFC is half a right angle; and since each Asymptote makes the same angle with the axis, the angle between the Asymptotes is a right angle.

Again, in the equilateral Hyperbola all conjugate diameters are equal, and the Asymptotes bisect the angle between any pair of conjugate diameters.

For let CP, CV be any pair of conjugate diameters.

Then (by Art. 135) $CP^2 - CV^2 = CA^2 - CB^2$

Therefore $CP^2 = CV^2$ and $CP = CV$.

Hence if the parallelogram CPHV be completed it is a rhombus and the diagonal CH which is an Asymptotes bisects the angle PCD.

Hence also supplemental chords are equally inclined to the Asymptotes, for they are parallel to conjugate diameters.

It is easy to see also that diameters at right angles to one another are equal.

Art. 144.

FIG. 78. If the normal at P meet the axes in G and g, then $CM = MG$ and $PG = Pg = CP$.

For (Art. 120) $MG : CM = BC^2 : AC^2 \therefore MG = CM$

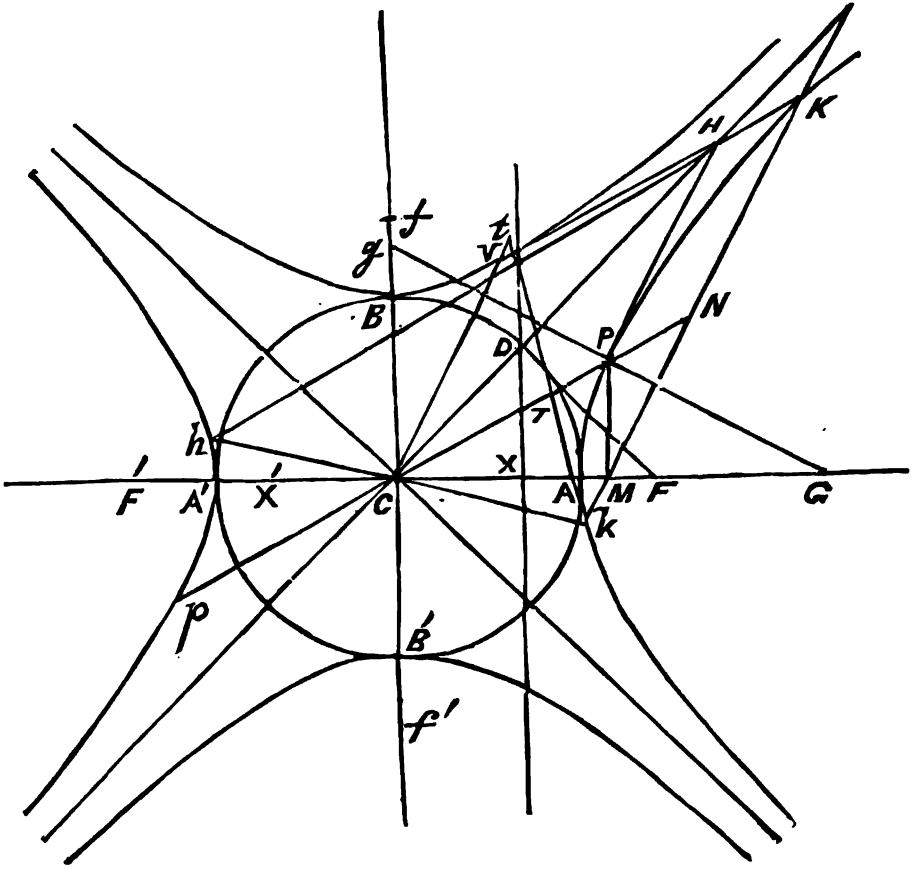
And since PM is parallel to CB,

Therefore $PG : Pg = MG : MC$

Therefore $PG = Pg = CP$, since GCg is a right angled triangle.

Further (by Art. 136) $PG.Pg = CV^2$, therefore $PG = Pg = CV$.

Fig. 78.



Art. 145.

Fig. 78. If KNk be an ordinate to CP, then $PN.Np = NK^2$.

For (by Prop. VII) $PN.Np : NK^2 = CP^2 : CV^2$

∴ (By Art. 144) $PN.Np = NK^2$.

Art. 146.

Fig. 78. If the tangent at any point k intersect a pair of conjugate diameters CP, CV in T and t, then $Tk.kt = Ck^2$.

For since conjugate diameters are equally inclined to the Asymptotes, the angle between any pair of diameters is equal to that between their conjugates; hence since the tangent kTt is parallel to the diameter conjugate to Ck, therefore the angle Ctk = TCk, and therefore the triangles Ctk, kCT are similar; hence $tk : Ck = Ck : Tk$, and therefore $Ck^2 = Tk kt$

It is also evident that the angles PCk, TkK are equal, for tkK = Ctk = PCk.

Hence if kC be produced to meet the opposite branch of the curve in h, and Kh be joined, then the angle Khk = TCk = TkK.

Art. 147.

FIG. 79. **In an equilateral Hyperbola, any chord subtends at the ends of any diameter angles which are either equal or supplementary.**

Let Pp be any diameter of an equilateral Hyperbola and QR any chord, then supposing Q to remain fixed, let R have successively the positions indicated in the figure, and in each position join Q and R with P and p.

Draw the tangents LPL′, lpl′, then taking the chord QR.

By the last Art. the angle $LPR = RpP$
Also $LPQ = QpP$
∴ By subtraction $QPR = QpR$.

Again, take the chord QR_1.

By Art. 146 the angle $LPQ = QpP$
Also $LPR = RpP$
∴ By subtraction $QPR = QpR$.

Again, take the chord QR_2.

By Art. 146 the angle $LPQ = QpP$
Also $L'PR_2 = R_2pP$
∴ By addition $LPQ + L'PR_2 = QpR_2$
Therefore QPR_2 and QpR_2 are supplementary.

Again, take the chord QR_3.

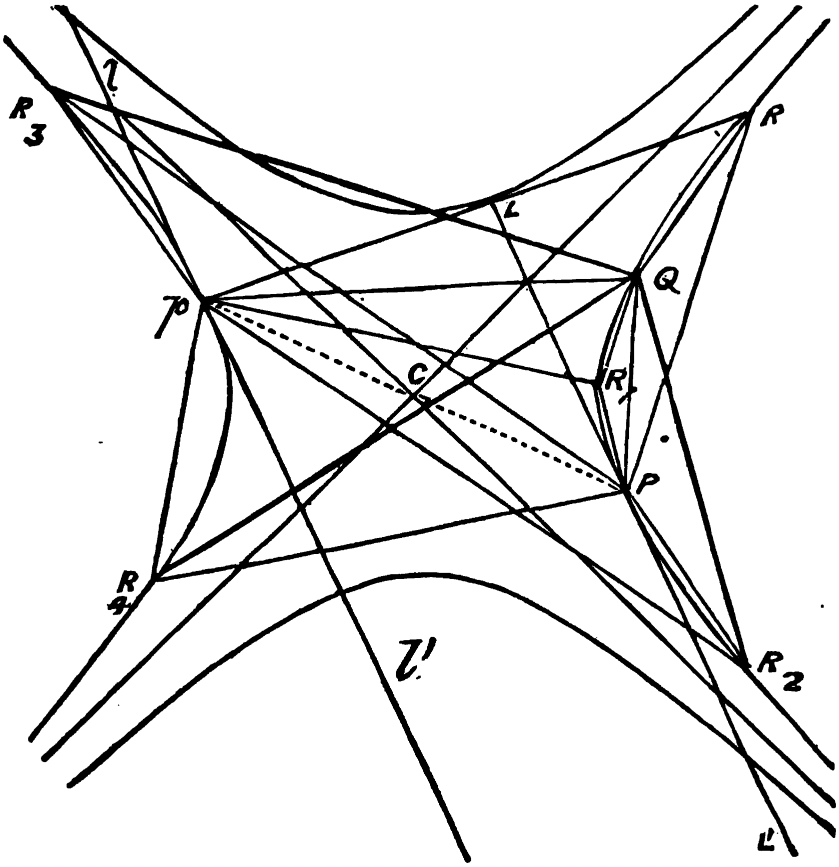
Then the angle $QPR_3 = QPL + LPR_3$
By Art. 146 $= QpP + LPR_3$
Also the angle $QpR_3 = Qpl + lpR_3$
By Art. 146 $= Qpl + R_3Pp$
Therefore by addition $QPR_3 + QpR_3 = LPp + lpP$
$= 2$ right angles
Therefore QPR_3 and QpR_3 are supplementary.

Again, take the chord QR_4.

By Art. 146 the angle $LPQ = QpP$
Also by the same Art. $R_4Pp = l'pR_4$
And by parallels $LPp = l'pP$
Therefore by addition $QPR_4 = QpR_4$.

Therefore the angles the chord QR subtends at the ends of any diameter are either equal or supplementary.

Fig. 79.



The Student can pursue the investigation further for himself by observing the change that takes place in the theorms of the general Hyperbola that have already been given when the curve becomes equilateral; at the end of the examples on the Hyperbola a number also of examples on the equilateral Hyperbola are given.

EXAMPLES ON THE HYPERBOLA.

1.—In an Hyperbola the tangents at the vertices, the Asymptotes and the circle on FF' as diameter, all pass through the same points.

2.—If the tangent at B to the conjugate Hyperbola meet the Latus Rectum in D, the triangles FCD, FXD are similar.

3.—If the tangent at P meet the directrices in ZZ', then $PZ.PZ' = CD^2$; CD, CP being conjugate diameters.

4.—Perpendiculars from the foci on the Asymptotes meet them on the auxiliary circle.

5.—The locus of the centre of the circle inscribed in the triangle FPF' is the tangent at the vertex.

6.—An Hyperbola and Ellipse having the same foci cut one another at right angles.

7.—If the tangents at P and Q intersect the tangents at A and A′ in R, r, S, s, prove that $AR.Ar = AS.As$.

8.—Given the base of a triangle and the point of contact of the base with the inscribed circle, find the locus of vertex.

9.—If PCp be a diameter and Fp meet the tangent at P in Z, prove that $FP = FZ$.

10.—Construct the Hyperbola, having given an Asymptote, directrix, and point of the curve.

11.—If through any point P of an Hyperbola, lines are drawn parallel to the Asymptotes and meeting any semi-diameter CK in Z and Z′, shew that $CZ.CZ' = CK^2$.

12.—If A and A′ be joined with any point on the curve and produced to meet any ordinate, prove that the rectangle of the segments of the ordinate between the axis and the lines drawn from A and A′ is equal to the square of the ordinate.

13.—If the tangent and focal distance produced of any point P on an Hyperbola cut any Asymptote in T and Z, shew that $FZ = ZT$.

14.—If PN be an ordinate to the axis, prove that $CP^2 = CA^2 - CB^2 + FP.PF'$

15.—If the normal at P meet the axis in G, then $PG^2 = FG.GF' - FP.PF'$.

16.—If PN be an ordinate to the axis, and NQ be draw parallel to AB joined, then

$$PN^2 = CQ^2 - CB^2 = BQ\,QB'.$$

17.—If the tangent from P intersect the axes in T, and Y, Y′ be the feet of perpendiculars on the tangent from the foci, then $Y'P : PY = Y'T : TY$.

18.—If YY′ be the feet of perpendiculars from the foci on the tangent at P, and CB be the semi-axis minor, then $F'P : PF = F'Y'^2 : CB^2 = CB^2 : FY^2$.

19.—If the perpendiculars to the axis from A, C, A′ meet the tangent at P in K, G, M, and the ordinate PN be drawn, then AK : PN = CG : A′M.

20.—If PQ be any chord of an Hyperbola, and V the point of contact of the parallel tangent, and PL, VM, QN be drawn parallel to one Asymptote and meeting the other in L, M, N, prove that $CL.CN = CM^2$.

21.—Show that the tangent to an Hyperbola forms with Asymptotes a triangle of constant area.

22.—If CP, CD be conjugate semi-diameters, and through C a line be drawn parallel to either focal distance of P, the perpendicular from D on this line is equal to the semi-axis minor.

23.—The difference of two focal chords which are parallel to conjugate diameters of an Hyperbola is constant.

24.—Determine the direction of the axes, having given a pair of conjugate diameters.

25.—Given a point on an Hyperbola and the Asymptotes, find the foci and construct the curve.

26.—The rectangles under the segments of straight lines drawn through any point to meet the Asymptotes are to one another as the squares of the parallel semi-diameters.

27.—If from two points in a Hyperbola straight lines be drawn to the centre, the area of the sector thus formed is equal to the trapezium formed by drawing from the points parallels to either of the Asymptotes.

28.—The part of the Latus Rectum produced which is between the curve and an Asymptote is equal to the radius of the circle which touches the Hyperbola and its Asymptotes.

29.—If AA′ be any diameter of a circle, and PNQ an ordinate to it, then the locus of the intersections of AP, A′Q is a rectangular Hyperbola, find its axes and foci.

30.—If from one focus of a Hyperbola a circle with radius equal to BC be described, the chord of contact of tangents drawn to it from any point in the other branch of the Hyperbola will touch the auxiliary circle.

31.—If through two points on an Hyperbola lines be drawn parallel to both Asymptotes, and thus forming a parallelogram of which the lines joining the points forms one diagonal, the other diagonal passes through the centre.

32.—If a straight line move so as to cut off a constant area from the corner of a rectangle, the locus of the middle point of the line is an equilateral Hyperbola.

33.—The focal distance of any point on an Hyperbola is equal to the ordinate of the point produced to meet the tangent at the extremity of the Latus Rectum, prove this.

34.—Let tangents be drawn at A and P the vertices of the axis and any other diameter meeting their diameters in E and T respectively, then the triangles CAE, CTP are equal.

35.—If TP, TQ touch an Hyperbola in P and Q, and PQ be joined, and any line be drawn parallel to TQ cutting TP, PQ in G and H, and the curve in R and L, then GR, GH, GL are continually proportional.

36.—If a tangent to a Hyperbola cut a diameter CP in T and its conjugate in G, and also cut the tangents at the extremities of PCp in K and H, then PK, CD, Hp are in continued proportion.

37.—Any line drawn through the intersection of a pair of tangents to meet the curve and chord of contact is divided harmonically by the curve, chord of contact, and the point of concourse of the tangents.

38.—If lines be drawn from the foci of an Hyperbola to two points on the same side of the axis of one branch of the curve, then a circle may be described touching the four focal distances.

39.—The straight lines joining each focus to the foot of the perpendicular from the other focus on the tangent meet on the normal and bisect it.

40.—If the tangent and normal at P meet the axis in T and G, then $NG.CT = BC^2$.

41.—If the tangent at P meet any conjugate diameters in T and t, the triangles FPT, F′Pt are similar.

42.—If the diameter conjugate to CP meet FP, F′P in E and E′, then the circles about the triangles FCE, F′CE′ are equal.

43.—The locus of the centre of the circle inscribed in the triangle FPF′ is a straight line, prove this.

44.—If G be the foot of the normal, and if the tangent meet the Asymptotes in R, S, then

$$GR = GS.$$

45.—If a circle touch the transverse axis at a focus and pass through one end of the conjugate, the chord intercepted by the conjugate is a third proportional to the semi-axes.

46.—If PFQ be a focal chord, and if the tangents at P and Q meet in T, the difference between PTQ and half PF′Q is a right angle.

40.—Any diameter is a mean proportional between the parallel focal chord and the transverse axis.

48.—If any two tangents be drawn to an Hyperbola and their intersections with the Asymptotes joined, the joining lines will be parallel.

49.—If any chord AP through the vertex be divided in Q, so that $AQ : QP = AC^2 : BC^2$, and QN be drawn to the foot of the ordinate PN, prove that a straight line drawn at right angles to QN from Q cuts the transverse axis in the same ratio.

50.—P and P′ are the extremities of conjugate diameters on an Hyperbola and its conjugate, show that $fP - FP = AC - BC$, F and f being the interior foci.

51.—If two parallel lines be drawn cutting an Hyperbola and meeting a line drawn through any point on the curve parallel to an Asymptote, the rectangles of the segments of the parallels are to one another as the segments they intercept upon the line parallel to the Asymptote measured from the point through which it is drawn; also in like manner the squares of parallel tangents (between their points of contact and any parallel to an Asymptote) are to one another in the same ratio.

52.—If from two points, one in an Hyperbola and the other in the conjugate Hyperbola, two lines be drawn parallel to either of the Asymptotes, the rectangles contained by the said lines and the segments (measured from the centre) they cut off on either of the Asymptotes are equal to one another.

53.—If through any point on an Hyperbola two right lines parallel to the Asymptotes be drawn meeting a diameter, the square of the semi-diameter is a mean proportional between its segments between the centre and the parallels to the Asymptotes.

54.—If two tangents to an Hyperbola be cut by a third tangent, the rectangle of the segments of the parallel tangents between their points of contact and the third tangent is equal to the square of the semi-diameter to which they are parallel; also the rectangle of the segments of the third tangent between its point of contact and the parallel tangents is equal to the square of the semi-diameter to which it is parallel.

55.—Show that the points of trisection of a series of conterminous circular arcs lie on branches, of two Hyperbolas, determine the distance between their centres.

56.—If a parallel to an Asymptote be drawn cutting a pair of tangents to an Hyperbola, the curve and their chord of contact, show, that the rectangle of its segments between the curve and the tangents is equal to the square of its segments between the curve and chord of contact.

57.—If through any two points either on the same or on opposite branches of an Hyperbola two lines be drawn to any third point on either branches of the curve and meeting a parallel to an Asymptote, the segments of this parallel between the curve and the two lines first drawn shall have to one another the same ratio wherever on the curve the third point be taken.

58.—The distance of any point on an Hyperbola from one of the foci is equal to the ordinate at the same point produced to meet the tangent drawn through the extremity of the focal ordinate.

59.—If P any point on an Hyperbola be joined with the focus F, and PN be drawn parallel to any Asymptote and meeting the directrix nearest to F in N, prove that PF $=$ PN.

60.—If any third tangent to an Hyperbola meet the tangents at the extremities of the transverse axis in G and H, prove that the circle on GH as diameter passes through the foci.

61.—If CK be a perpendicular from the centre of an Hyperbola on the tangent at P, and PN a perpendicular on the transverse axis, show that PN.CK $=$ square of the semi-axis minor.

62.—In an Hyperbola any diameter is a mean proportional between the chord drawn through either focus (and meeting either one or both branches of the curve as the case may be) and the transverse axis.

63.—If FY be drawn perpendicular to the tangent at P, and FP be joined, then FY is to FP as the semi-axis minor to the semi-diameter parallel to the tangent.

64.—If between a rectangular Hyperbola and its Asymptotes any number of concentric elliptic quadrants be inscribed, the rectangle contained by their axes will be constant.

65.—PP' is a double ordinate of an Ellipse, AP, A'P' are produced to meet in Q, shew that the locus of Q is an Hyperbola, and find its axes and foci.

66.—If tangents at P and P' of an Hyerbola be drawn meeting in N, and the diameters CP, CP' also be drawn meeting the tangents in T and T', then the triangle P'NT' = PNT. Show that this is true when the tangents touch opposite branches of the curve.

67.—If QH be a tangent at any point Q; PC, VC conjugate diameters cutting the tangent in Z and H, then if CR be the diameter conjugate to CQ, the rectangle ZAH = CR^2.

68.—If PT, QT be two tangents at P and Q, and PQ be joined, then if any line cuts PQ and the curve in S, A, B and PT, QT in L, D, then $SL^2 : SD^2 = AL.LB : BD.DA$.

69.—If tangents at P, p, the extremities of any diameter be cut by a third tangent in T and T', and CV be the diameter parallel to the third tangent the point of contact of which is N, then $TN.NT' = CV^2$.

70.—If two tangents HE, HK touching the conjugate Hyperbolas in R and D cut any diameter AB in E and K so that AE = BK; and if TZ be a third tangent touching the curve in I and cutting the former tangents in T and G, then the rectangle ET.GK = KH.RE a given quantity.

71.—If ABFE be an inscribed trapezium, the line joining the middle points of its diagonals AF, BE will pass through the centre.

72.—If from the extremites of any diameter straight lines be drawn to any point Q of an Hyperbola, and an ordinate PN to the be drawn meeting the lines in the points G and E, then NE, NP, NG are in continued proportion.

73.—If CQ be conjugate to the normal at P, then is CP conjugate to the normal at Q.

74.—Find the locus of the points of contact of tangents to a series of confocal Hyperbolas from a fixed point in the axis.

75.—Tangents to an Hyperbola are drawn from any point in one of the branches of the conjugate, show that the chord of contact will touch the other branch of the conjugate.

76..—An Hyperbola of given excentricity always passes through two given points; if one of the Asymptotes always pass through a third given point in the same straight line with these, the locus of the centre of the Hyperbola will be a circle.

77.—An ordinate NP meets the the conjugate Hyperbola in Q, prove that the normals at P and Q meet on the transverve axis.

78.—Having given the focus and three points on an Hyperbola to describe the curve.

79.—To describe an Hyperbola about an given parallelogram and so as to pass through a fixed point P.

80.—If an Hyperbola be described touching the four sides of a quadrilateral which is inscribed in a circle and one focus lie on the circle, the other focus will also lie on the circle.

81.—If PN be the ordinate of a point P, and PT the tangent meeting the axis in T, then

$$FP : FT = AN : AT.$$

82.—The tangent and normal at any point intersect the Asymptotes and axes respectively in four points, which lie on a circle passing through the centre of the curve.

83.—A Conic section is drawn touching the Asymptotes of an Hyperbola, prove that two of the chords of intersection of the curves are parallel to the chord of contact of the Conic with the Asymptotes.

84.—Show that the locus of the intersections of tangents to an Hyperbola that intersect at right angles is a circle concentric with the Hyperbola, and the square of whose radius $= CA^2 - CB^2$.

RECTANGULAR HYPERBOLA.

85.—If a rectangular Hyperbola circumscribe a triangle it passes through the centre or intersection of perpendiculars from the angles on the sides.

86.—If a rectangular Hyperbola circumscribe a triangle, the locus of the centre is the nine point circle of the triangle.

87.—PCp is a transverse diameter and QV an ordinate, show that QV is the tangent at Q to the circle circumscribing the triangle PQp.

88.—If the tangent at P meet the Asymptotes in L and M, and the normal meet the transverse axes in G, a circle can be drawn through C, L, M and G, and LGM is a right angle.

89.—In a rectangular Hyperbola it is not possible to draw a pair of tangents at right angles to another.

90.—The locus of the centre of an equilateral Hyperbola described about a given equilateral triangle is the circle inscribed in that triangle.

91.—The distance of any point from the centre is a geometric mean between its distances from the foci.

92.—Straight lines drawn from any point on the curve to the extremities of a diameter are equally inclined to the Asymptotes.

93.—In a rectangular Hyperbola focal chords parallel to conjugate diameters are equal.

94.—The distance of any point from the centre is a geometrical mean between its distances from the foci.

95.—The tangents at the vertices of the axis intersect the Asymptotes on the circumference of the circle on FF′ as diameter.

96.—If the tangents at two points Q and Q′ meet in T, and if CQ, CQ′ meet these tangents in R and R′, the circle circumscribing RTR′ passes through C.

97.—Q is a point on the conjugate axis of a rectangular Hyperbola, and QP drawn parallel to the transverse axis meets the curve in P, prove that PQ $=$ AQ.

98.—Given an Asymptote and a tangent at a given point, construct a rectangular Hyperbola.

99.—Focal chords parallel to conjugate diameters are equal.

100.—If the tangent at Q meet CP in T, the circle circumscribing CTQ touches the ordinate QV conjugate to CP.

101.—Every Conic passing through the centres of the four circles which touch the sides of a triangle is a rectangular Hyperbola.

102.—The foci of Ellipses inscribed in a given parallelogram lie on a rectangular Hyperbola.

103.—If a right angled triangle be inscribed in the curve, the normal at the right angle is parallel to the hypothenuse.

104.—On opposite sides of any chord of a rectangular Hyperbola are described equal segments of circles, show that the four points in which the circles to which these segments belong again meet the Hyperbola are the angular points of a parallelogram.

105.—The tangent at a point P of a rectangular Hyperbola meets a diameter QCQ′ in T, show that CQ and TQ′ subtend equal angles at P.

106.—In the rectangular Hyperbola prove that the triangle formed by the tangent at any point and its intercepts on the axes is similar to the triangle formed by the straight line joining that point to the centre and the abscissa and ordinate of the point.

107.—P is a point on a rectangular Hyperbola, and PM, PN are straight lines making equal angles with one of the Asymptotes, if MP, NP be produced to meet the curve in P′ and Q′, then P′Q′ passes through the centre.

108.—Straight lines parallel to conjugate diameters meet the Asymptotes in four points which lie on a circle.

109.—If two rectangular Hyperbolas touch one another, their common chords through the point of contact will include a right angle, and the remaining common chord will be parallel to the common tangent.

110.—The part of the tangent intercepted by the Asymptotes subtends a right angle at the foot of the normal.

111.—Any Conic which passes through the four points of intersection of two rectangular Hyperbolas must be itself a rectangular Hyperbola.

112.—Given a triangle such that any vertex is the pole of the opposite side with respect to an equilateral Hyperbola, the circle circumscribing the triangle passes through the centre of the curve.

113.—A circle described through the centre of a rectangular Hyperbola and any two points will also pass through the intersection of lines drawn through each of these points parallel to the polar of the other.

114.—If a rectangular Hyperbola having its Asymptotes coincident with the axes of an Ellipse touch the Ellipse, the axis of the Hyperbola is a mean proportional between the axes of the Ellipse.

115.—If two focal chords be parallel to conjugate diameters, the lines joining their extremities intersect on the Asymptotes.

116.—Two lines of given lengths coincide with and move along two fixed lines in such a manner that a circle can always be drawn through their extremities, the locus of the centre is a rectangular Hyperbola.

117.—The locus of the middle point of a line which cuts off a constant area from the corner of a given square is a rectangular Hyperbola.

118.—If the base of a triangle be given in position and magnitude, and also the difference of the angles at the base be given, the locus of the vertex is an equilateral Hyperbola given in position—apply this theorm to trisect a given rectilineal angle.

SIMILAR CONICS.

Art. 148.

DEF.—Conics which have the same eccentricity are said to be similar to each other.

From this definition it follows at once that all Parabolas are similar figures.

Let CA, CB be the semi-axes of one Ellipse or Hyperbola, and ca, cb those of another similar to the first, also let CR, cr be two radii making equal angles with the axes CA, ca.

If then F and f be foci, since the curves are similar the eccentricities are the same. (By the def.)

Therefore $AC : CF = ac : cf$

And ∴ $AC^2 : AC^2 - CF^2 = ac^2 : ac^2 - cf^2$

Or $AC^2 : BC^2 = ac^2 : bc^2$ (1)

But if PN, pn be ordinates to the axes AC, ac,

$AC^2 - CN^2 : PN^2 = AC^2 : BC^2$

And $ac^2 - cn^2 : pn^2 = ac^2 : bc^2$

∴ Ex æquali $PN^2 : AC^2 - CN^2 = pn^2 : ac^2 - cn^2$

But by sim. tri. $PN : pn = CN : cn$

∴ $CN^2 : AC^2 - CN^2 = cn^2 : ac^2 - cn^2$

And $CN^2 : AC^2 = cn^2 : ac^2$ (2)

But by sim. tri. $CP : cp = CN : cn$

∴ $CP : cp = CA : ca.$ (3)

By (1) it appears that similar Ellipses and Hyperbolas have their axes in the same proportion, and by (3) it appears that in the same curves the diameters that make equal angles with the axes are to one another as the axes.

By (2) $AC : CN = ac : cn$

∴ Div^{do} et alt^{do} $AN : an = CN : cn$

And Comp^{do} et alt^{do} $A'N : a'n = CN : Cn$

But by sim. tri. $PN : pn = CN : cn$

Therefore $PN : pn = AN : an = A'N : a'n.$

That is, in similar Ellipses and Hyperbolas the ordinates are to one another as the abscisas cut off from either end of the axes; this result Apollonius adopted as the criterion of similarity in his work on the Conics. (See Ap., Bk. 6, Def. 2.)

*The preceding proof applies both to the Ellipse and Hyperbola, but in the case of the latter curve a simpler test of similarity furnished by the consideration that if in two Hyperbolas $AC : CF = ac : cf$, the angles between their Asymptotes are equal, and therefore conversely if the angles contained by the Asymptotes of two Hyperbolas are equal the curves are similar.

---

* The demonstration above given is applicable to the Ellipse, but the reader will easily adapt to that of the Hyperbola.

Art. 149.

If the extremities of two semi-diameters that make equal angles with the axis in two similar Conics be joined with the vertices of the axis, the joining lines are always to one another as the axes.

For from what has been proved the triangles PAC, pac are similar, therefore

$$AP : ap = AC : ac$$

And the converse of this is also true, viz.: if in two similar Conics radii be drawn from the vertices of the axes making equal angles therewith they are to one another in the constant ratio of the axes.

In the case of the Parabola if AP, ap be the radii, PN and pn the ordinates, the angles PAN, pan being equal.

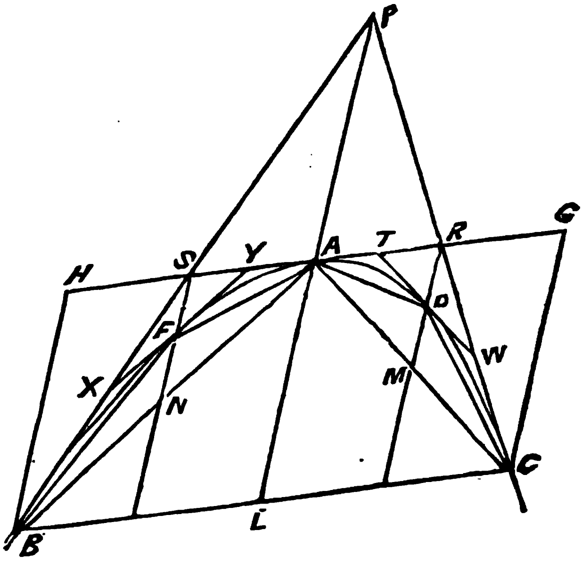
By sim. tri. $AN : an = AP : ap$
And $PN : pn = AP : ap$
But $PN^2 : pn^2 = AF.AN : af.an$
$\therefore$ $AP^2 : ap^2 = AF.AP : af.ap$
Or $AP : ap = AF : af.$

That is, the radii from the vertices making equal angles with the axes are to one another as the Latera Recta of the Parabolas.

And generally it may be proved that lines drawn similarly from the foci or any other corresponding points of similar Conics are to another as the transverse axes of the curves.

AREAS OF THE CONIC SECTIONS.

Fig. 80.



Art. 150.

THE PARABOLA.

Fig. 80. Let BAC be a segment of a Parabola contained by the Parabolic arc BAC and the straight line BC.

From the points B and C draw the tangents BP, CP meeting in P, and draw the diameter PAL cutting the curve in A, draw the tangent SAR meeting BP, CP in S and R, and join BA, CA, then by (Art. 72) PA = AL, and therefore BC = 2SR, and therefore the triangle BAC = 2SPR.

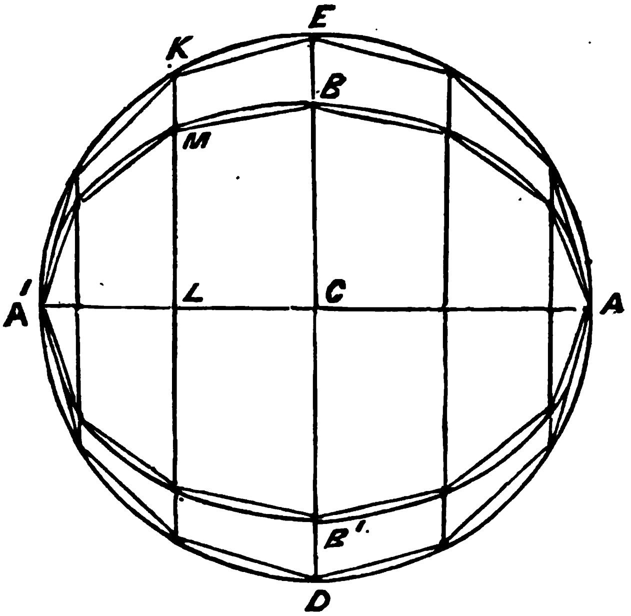
Again, if the diameters RDM, SEN be drawn and the tangents WDT, XEY, and CD, AD, BE, AE be joined, it may be shown in the same manner that the triangle ADC = 2TRW, and the triangle BEA = 2XSY. Continuing this process by drawing diameters through the points W, T, Y, X, &c.

The sum of all the interior triangles BAC, BEA, ADC, ... is double of the sum of the exterior triangles SPR, XSY, WRT, ... and since the triangles both interior and exterior, viz., BAC, BEC, ADC, ... SPR, XSY, TRW .... are greater than half the corresponding parabolic segments, viz., BAC, BEC, ADC, ... PBAC, SBEA, RCDA.... therefore (by Eu. Bk. 12) the differences between the sum of the interior triangles and the Parabolic segment BAC and the sum of the exterior triangles and area PBAC can be made less than any assignable quantities; therefore the area of the segment BAC is double of the exterior area contained by the tangents BP, PC and the Parabolic arc BAC.

Cor.—Complete the parallelogram BCGH, then since PL = 2LA, the parallelogram BCGH is evidently equal to the triangle BPC; but the Parabolic segment BAC is two-thirds of the triangle BPC as has been proved, therefore the segment BAC is two-thirds of its circumscribing parallelogram BCGH.

Art. 151.

Fig. 81.



THE ELLIPSE.

Fig. 81.

Let ABA'B' be an Ellipse of which AA' is the diameter, on AA' as diameter describe a circle AEA'D, in this circle inscribe a polygon of any number of sides, and from each of the angular points E, K, &c., draw perpendiculars to AA' cutting the Ellipse in the points B, M, &c., then since each of the lines EC, KL, &c., is divided in the same ration, viz., that of AC to BC in the points B, M, &c. It is evident that the trapezium ECLK has to the trapezium BCLM the same ratio of AC to BC, and the same is true of every other pair of inscribed trapezia, and also of the triangles at the points A and A'.

If therefore the number of sides of the inscribed polygons be made indefinitely great, the polygons will coincide with the circle and Ellipse respectively, and the circle will be to the Ellipse as AC to BC.

Cor.—Hence the Ellipse is equal to a circle the diameter of which is a mean proportional between its axes.

For the circle is to the Ellipse as $AC : BC$

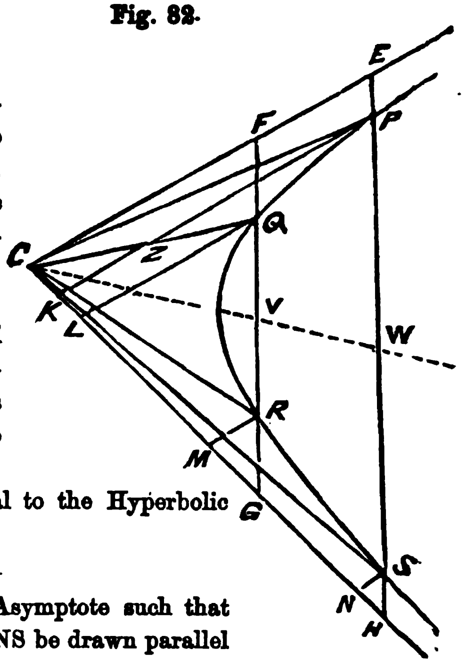
That is as $AA' : BB'$

" $AA'^2 : AA'.BB'$.

" $AA'^2 : Z^2$.

If Z be a mean proportional between AA' and BB'; but as AA'^2 to Z^2 so is the circle on AA' to that on Z as diameter; therefore the Ellipse is equal to the circle whose diameter in Z, that is a mean proportional between its axes AA' and BB'.

Fig. 82.



Art. 152.

THE HYPERBOLA.

Fig. 82. If P and Q be two points upon a Hyperbola, and PK, QL be drawn parallel to one Asymptote and meeting the other in K and L, and CP, CQ be joined, then the Hyperbolic sector PCQ is equal to the Hyperbolic trapezium PKLQ.

Let PK meet CQ in Z.

Then since the triangles CPK, CQL are equal (see Art. 141) taking away from each the common part CZK, and adding to the remainders the area PZQ, the sector PCQ is equal to the trapezium PKLQ.

That is, the Hyperbolic sector PCQ is equal to the Hyperbolic trapezium PKLQ.

Art. 153.

If four points K, L, M, N be taken on an Asymptote such that

Fig. 82. CK : CL = CM : CN, and if KP, LQ, MR, NS be drawn parallel to the other Asymptote and meeting the curve in P, Q, R, S, and CP, CQ, CR, CS be joined, the Hyperbolic sectors PCQ, RCS are equal.

Let PS, RQ produced meet the Asymptotes in E, H, F, G, then (by Art. 127) QF = RG and PE = SH, therefore MG = CL and NH = CK.

Therefore MG : NH = CL : CK
= CN : CM by hypothesis
= MR : NS by Art. 141

Therefore QR is parallel to PS.

Hence if CVW be the diameter conjugate to PS, QR, and therefore bisecting them and all chords parallel to them, the diameter CVW will bisect the area PQRS; and since the triangles PCW, SCW are equal, and also the triangles QCV, RCV.

Therefore taking away from the triangle PCW the area PQVW and the triangle QCV, and from the triangle SCW the area SRVW and the triangle RCV, the remaining sector PCQ is equal to the remaining sector RCS.

Cor.—If any number of points K, L, M, N be taken in one of the Asymptotes such that CK, CL, CM, CN .... are in continual proportion, it follows that the Hyperbolic sectors CPQ, QCR, RCS .... are all equal. It will also be observed that in this case the tangents at Q and R are parallel to PR and QS respectively, and similarly for the rest, and this is proved in the same way that QR was shown to be parallel to PS.

It further follows when CK, CL, CM, CN .... are in continual proportion that the Hyperbolic trapezia PKLQ, QLMR, RMNS .... are all equal for they are equal to the sectors CPQ, QCR, RCS .... by Art. 152; therefore if CK, CL, CM, CN .... are in geometrical progression, either the sectors CPQ, PCR, PCS .... or the trapezia PKLQ, PKMR, PKNS .... are in arithmetical progression.

Art. 154.

Fig. 83. By the aid of Algebra the preceding articles may be extended and rendered more complete as follows :—

Let MAM′ be a branch of an Hyperbola and CL, CL′ the Asymptotes and F the focus.

Draw AQ perpendicular to CA and meeting CL in Q; and AP parallel to CL′ and meeting CL in P, therefore CQ = CF and CP = PQ = PA.

In PL take any point m, and also the points m_1, m_2 m_3 m_4 ...... such that CP, Cm, Cm_1, Cm_2, Cm_3, Cm_4 ...... are in continual proportion; also draw CG at right angles to CL′ and lmn, $l_1m_1n_1$, $l_2m_2n_2$, $l_3m_3n_3$, $l_4m_4n_4$ ...... parallel to CL′ and meeting the curve and CG in the points n, n_1, n_2, n_3, n_4 .... l, l_1, l_2, l_3, l_4 ....

Put CP = PQ = PA = a and Cm = x, then we have

$$CP = a,\ Cm = x,\ Cm_1 = \frac{x^2}{a},\ Cm_2 = \frac{x^3}{a^2},\ Cm_3 = \frac{x^4}{a^3} \ldots\ldots Cm_r = \frac{x^{r+1}}{a^r}$$

$$AP = a,\ mn = \frac{a^2}{x},\ m_1n_1 = \frac{a^3}{x^2},\ m_2n_2 = \frac{a^4}{x^3} \ldots\ldots m_rn_r = \frac{a^{r+2}}{x^{r+1}}$$

$$Pm = x-a,\ mm_1 = \frac{x}{a}(x-a),\ m_1m_2 = \frac{x^2}{a^2}(x-a),\ m_2m_3 = \frac{x^3}{a^3}(x-a) \ldots\ldots m_{r-1}m_r = \frac{x^r}{a^r}(x-a).$$

Hence if AP produced meet CG in R, and CP : CR = 1 : k

$$Rl = k(x-a),\ ll_1 = \frac{kx(x-a)}{a},\ l_1l_2 = \frac{kx^2(x-a)}{a^2} \ldots\ldots l_{r-1}l_r = \frac{kx^r}{a^r}(x-a)$$

Now the area of the rectilinear trapezium $m_{r-1}m_rn_rn_{r-1}$ is equal to

$$\frac{l_{r-1}l_r\,(m_{r-1}n_{r-1} + m_rn_r.)}{2}$$

Or by substituting the above values, is equal to

$$\frac{kx^r(x-a)}{a^r} \cdot \frac{a^{r+1}}{2x^{r+1}}(x+a) = \frac{ka}{2x}(x^2-a^2)$$

And since this is true for all values of r, it follows that all the rectilinear trapezia are equal, each being equal to $\frac{ka\,(x^2-a^2)}{2x}$ and the Hyperbolic trapezia are equal by Cor of Art. 153, therefore the differences of the rectilinear and Hyperbolic trapezia, that is the Hyperbolic segments, the bases of which are the lines An, nn_1, n_1n_2 .... $n_{r-1}n_r$ are also equal.

Draw ns, n_1s_1, n_2s_2 .... n_rs_r parallel to CL and meeting AP, mn, m_1n_1 .... in s, s_1, s_2, s_3 ....

Fig. 83.

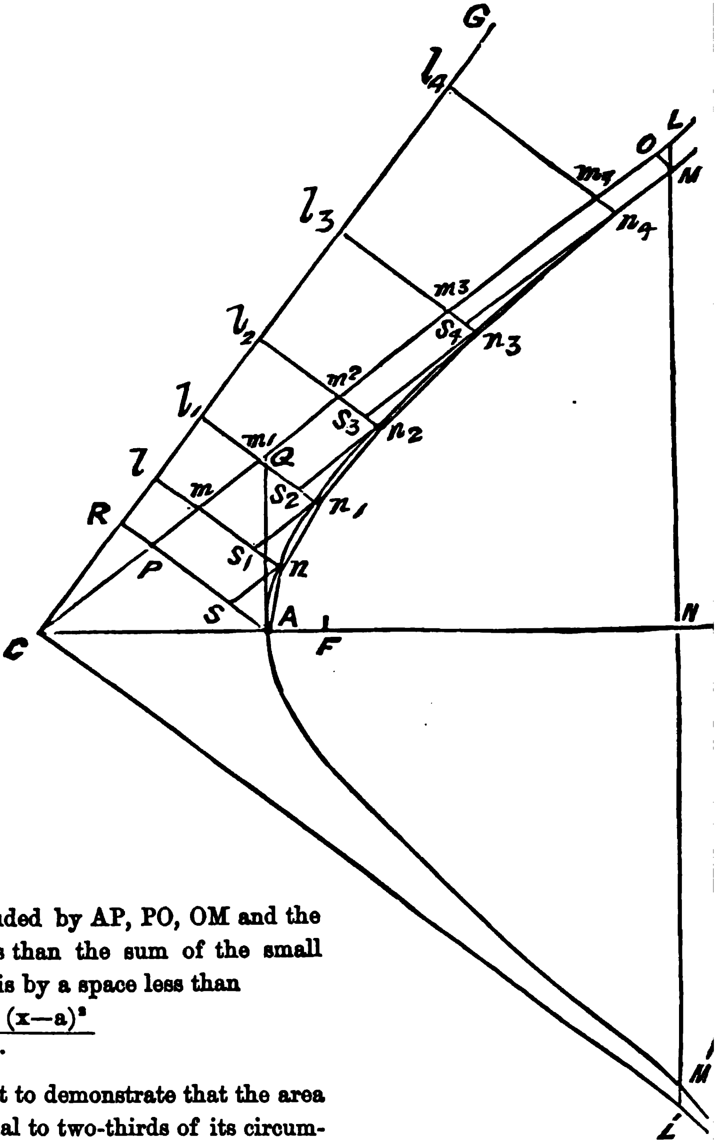


Fig. 83. Then the area of the small triangle $n_r s_r n_{r-1}$ is equal to

$$(m_{r-1}n_{r-1}-m_rn_r)\frac{l_{r-1}l_r}{2}$$

Which by substitution from the above values becomes

$$\left(\frac{a^{r+1}}{x^r}-\frac{a^{r+2}}{x^{r+1}}\right)\frac{kx^r(x-a)}{2a^r}=\frac{ka}{2x}(x-a)^2$$

Therefore the small triangles nsA, n_1s_1n, $n_2s_2n_1$ .... are all equal, each of them being equal to

$$\frac{ka(x-a)^2}{2x.}$$

Hence if MNM be an ordinate to the axis CAN of an Hyperbola, and MO be drawn parallel to CL′, and if between CO and CP there be inscribed r mean proportional, so that OC itself is the $\overline{r+1}^{th}$ proportional, then the whole rectilinear area $APOMn_rn_{r-1}n_{r-2}$.... A is equal to $\overline{r+1}$ of one of the trapeziums of which it is composed, that is to

$$\frac{(r+1)\,ka\,(x^2-a^2)}{2x.}$$

And this differs from the area included by AP, PO, OM and the Hyperbolic arc MA by a space less than the sum of the small triangles SnA, s_1n_1n ...... that is by a space less than

$$\frac{(r+1)\,ka\,(x-a)^2}{2x.}$$

Note.—Archimedes was the first to demonstrate that the area of a segment of a Parabola is equal to two-thirds of its circumscribing parallelogram, thus completely effecting the quadrature of the curve by geometrical means alone, though it appears from the preface to his treatise that he was led to the discovery by mechanical reasoning. No one, however, as yet has discovered any purely geometrical method for effecting the quadrature of either the circle, Ellipse, or Hyperbola, or their segments, but analytical methods have been given for ascertaining the areas of these curves with any required degree of accuracy

(See Wallace's Conic Sections, Bonnycastle's Mensuration, works on the Integral Calculus, &c.

ON THE CURVATURE OF THE CONIC SECTIONS.

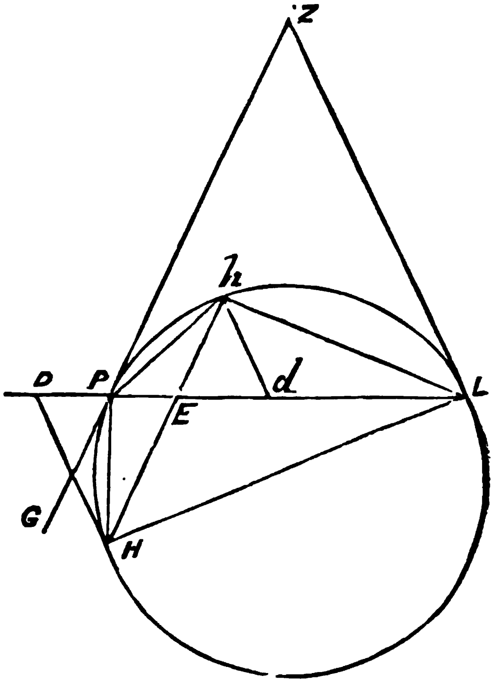
Art. 155.
Fig. 84. Def. 1.—A circle is said to touch a Conic section in any point, when the circle and Conic section have a common tangent at that point.

Def. 2.—If a circle touches a Conic section in any point so that no other circle touching it in the same point can pass between it and the Conic section on either side of the point of contact it is said to have the same curvature with the section at the point of contact, and is called the osculating circle of that point.

Lemma.—If HPL be a circle, PZ, ZL a pair of tangents thereto, and PL the chord of contact, then if from any point H in the circumference HD, HE be drawn parallel to ZL, ZP, then

$$HD^2 = HE^2 = LD.PE.$$

Fig. 84.



Join PH, HL, then since PZ = ZL, DH will equal HE; therefore the angle PEH = PDH = PLZ = PHL (by Eu. iii. 32); also the angles PHE, PLH are equal, for they are both equal to the angle GPH; therefore the triangles PHE, HLD are similar.

Therefore $LD : DH = HE (= HD) : PE$
And $LD.PE = HD^2 = HE^2$
Similarly it may be shown that $Ld.PE = hd^2 = hE^2$.

Art. 156.

PROB.—Having given a Conic Section, to draw the osculating circle at a given point P of section.

FIGS. 85 & 86. Draw the diameter of the section Pp, on which take PR equal to the parameter of the diameter Pp, draw the tangent PZ, then the circle touching the line PZ in the point P and passing through R will be the osculating circle required.

Draw the tangent ZRQ to the circle, and the tangent pQ to the Conic meeting in Q, and join PQ which will cut the circle again in one point K, draw KT, KS parallel to RZ, PZ respectively, then by parallels

$Pp : pT = PQ : QK = PR : RS$
Alternately $Pp : PR = pT : RS$
$= pT.PT : PT.RS$
But by the Lemma $PT.RS = KT^2$
Therefore $Pp : PR = pT.PT : KT^2$
Therefore the point K is on the section and KT is a semi-ordinate to the diameter Pp.

Again, draw any ordinate NEn to the diameter Pp meeting PQ in Y and the circle in H and h, also draw YV, HD parallel to RZ and meeting Pp in V and D.

Then by parallels $Pp : pE = PQ : QY = PR : VR$
Alternately $Pp : PR = pE : VR$
$= PE.Ep : PE.VR$
But $Pp : PR = PE.Ep : EN^2$
Therefore $EN^2 = PE\,VR$
And by the Lemma $EH^2 = PE.RD$
Therefore $EN^2 : EH^2 = VR : RD$
Similarly $En^2 : Eh^2 = VR : Rd$

And it is evident that if the ordinate NEn be supposed to move parallel to itself, that whilst H falls between K and P, RD will always be greater than VR and Rd less than VR, therefore the point H and the circular arc PHK will fall without the section and the point h, and the arc PhK entirely within the section.

Also no other circle touching the line PZ at P can be drawn so as to fall between the section and the circle PRK, for if it be possible let another circle PlX meeting the ordinate En in l fall between them on either side of P, and cut off from the diameter Pp a segment either greater or less than PR, and first let it be greater, then if lb be drawn parallel to the tangent of the circle at X, by the Lemma, $Xb.PE = El^2$

Therefore $En^2 : El^2 = RV : Xb$
Similarly $EN^2 : EL^2 = RV : XB.$

Fig. 85.

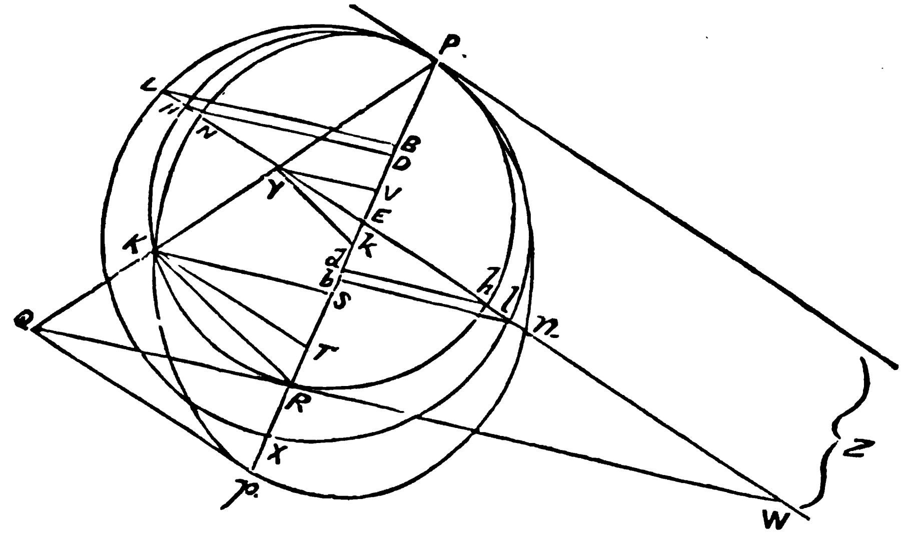
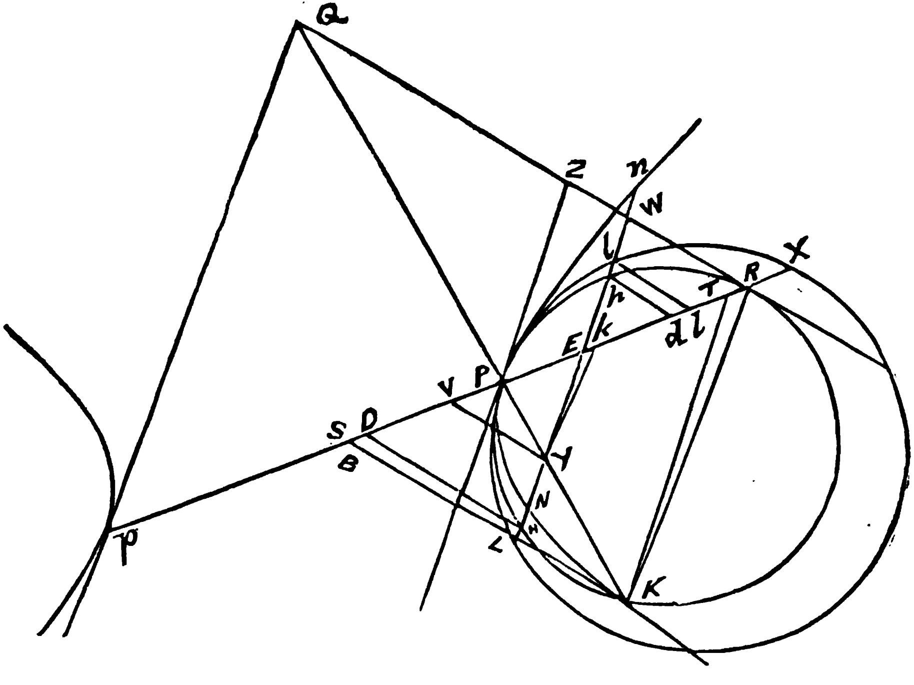


Fig. 86.



Now if the ordinate NEn move parallel to itself and approach the tangent PZ, the points b, V approach each other and coincide with P when E arrives at P, therefore there must be some position of E, for which bV $=$ RX, and therefore Xb $=$ RV; consequently in this position of the ordinate NEn, the points l and n coincide, and the circle and the section cut one another, therefore part of the circle nearer to P will lie without the section, also the arc PL lies without the circle PHK and therefore without the section. In the same way it may be shown that if PX is less than PR, there is a position of the ordinate NEn such that XB $<$ VR; and in this position and in all positions nearer to P, the point L of the circle falls within the section; but since in this case also the arc PlX lies within the circle PRK which is itself within the section, therefore whether PX be greater or less than PR, the circle of which PX is a chord cannot fall between the circle PRK and the section on either side of their point of contact, that is, no circle can fall between them on both sides of their point of contact; therefore PRK is the osculating circle of the point P that was to be drawn.

FIG. 87. The above demonstration is suited to the Ellipse and Hyperbola, but may easily be adapted to the Parabola. For draw PK parallel to the tangent RZ and meeting the circle in K, also draw KT parallel to the tangent PZ, then since by the Lemma PT.PR $=$ PK^2, and PR is the parameter of the diameter PE, therefore the point K is on the Parabola and KT is an ordinate to the diameter PE, and the rest follows as before.

Art. 157.

FIGS. 85, & 86.

The chord of contact PK and the tangent PZ are equally inclined to the axis.

For join KR, and draw Yk parallel thereto and meeting PR in k, then

By similar triangles KT : PT $=$ YE : PE

Also SK ($=$ KT) : SR $=$ YV (YE) : Vk

$\therefore$ KT^2 : PT.SR $=$ YE^2 : PE.Vk

But KT^2 $=$ PT.SR, therefore YE^2 $=$ PE.Vk

And NE^2 $=$ PE.VR, therefore NY.Yn $=$ NE^2$-$YE^2 $=$ PE.Rk

But by similar triangles YK : kR $=$ PE : PY

Therefore YK.PY $=$ kR.PE $=$ NY.Yn

And the rectangles YK.PY, NY.Yn are to one another as the squares of the parallel semi-diameters, therefore the parallel semi-diameters are equal and are therefore equally inclined to the axis;*consequently the chord of intersection KP and the tangent PZ are also equally inclined to the axis.

In the Parabola (fig. 87) since PK $=$ KT, the angle KPT $=$ PTK $=$ TPZ, therefore PK and PZ are both equally inclined to the diameter PE and consequently to the axis also.

The above corollary agrees with and may be deduced from Prop. xi., for if in that proposition the points P and Q coincide so that O'P becomes a tangent at P, then if one of the remaining points as R move up to and coincide with P also, the line SP becomes the chord of intersection and is equally inclined to the axis with the tangent at P.

---

* See note at foot of page 38.

Fig. 86.

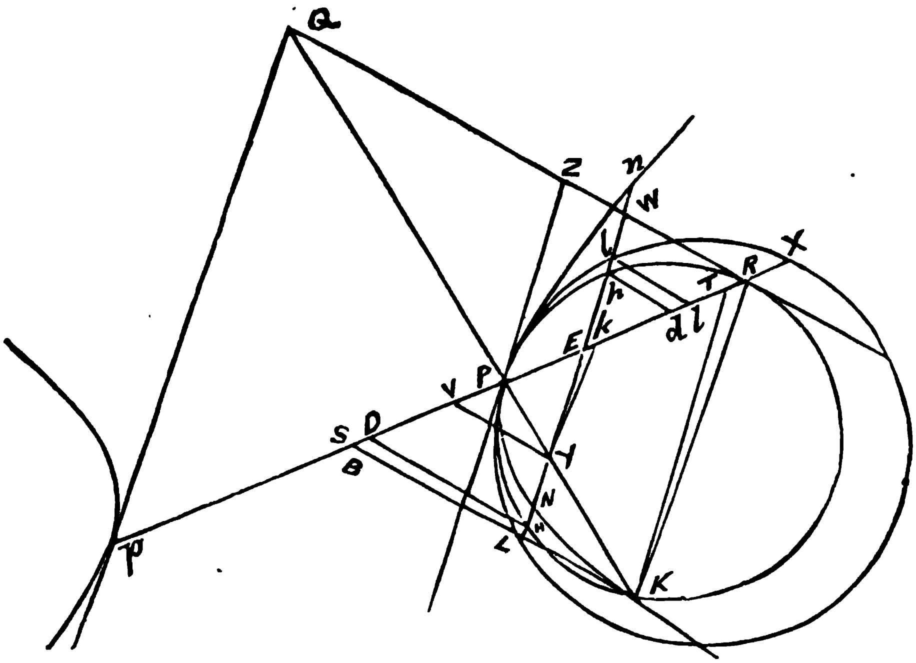
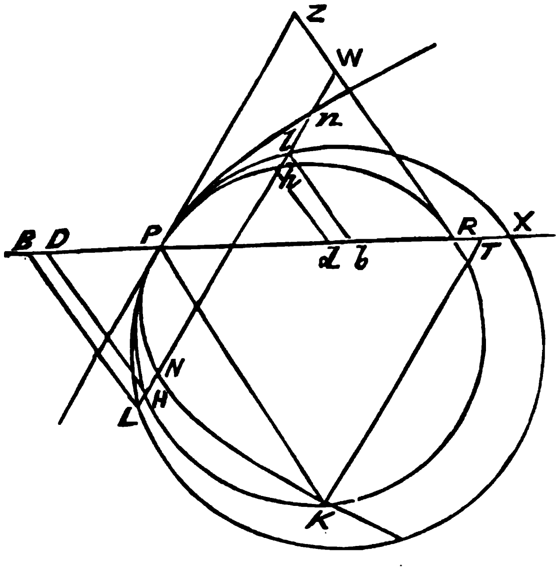


Fig. 87.



Art. 158.
Fig. 88. In the Parabola if PK be the chord of intersection of the osculating circle of the point P, and PT the intercept of the tangent between P and the axis, then

$$PK = 4PT.$$

For draw the ordinate PNp, and join Tp, and draw the diameter pE meeting PK in E.

Then since PK and PT make equal angles with the axis, viz., PTH and PHT, therefore $PT = PH$, and the angle $pTH = PTH = PHT$, therefore pT is parallel to PK; consequently PK is an ordinate to the diameter PE, (Prop. viii.)

Therefore $$PK = 2PE = 4PH = 4PT.$$

Art. 159.
Fig. 88. To find the chords of curvature through the focus and parallel to the axis, and the diameter of curvature at any point P of a Parabola.

Let F be the focus, and produce PF to meet the osculating circle in V, and let PK meet the axis in H, and join VK, then since the angle $FHP = FTP = FPT$ (by Art. 60) $=$ VKP (by Eu. iii. 32), therefore VK is parallel to the axis TH; hence $KP : PH = VP : PF$, but $KP = 4PH$, therefore $PV = 4FP$.

Also if PGO be the diameter of curvature and PR the chord parallel to the axis, since PG the normal bisects the angle FPR, (Art. 65) therefore $PV = PR$.

Join VO and draw FY perpendicular to PT, then the triangles POV, PFY are evidently similar, therefore $PO : PV\ (4PF) = PF : FY$.

Or thus, let GL be drawn perpendicular to PF, then (by Art. 55) PL is equal to the semi Latus Rectum.

And by sim. tri.
$$\begin{aligned} PO : PG &= PV : PL \\ &= 4FP : PL \\ &= 4FP.PL : PL^2. \end{aligned}$$

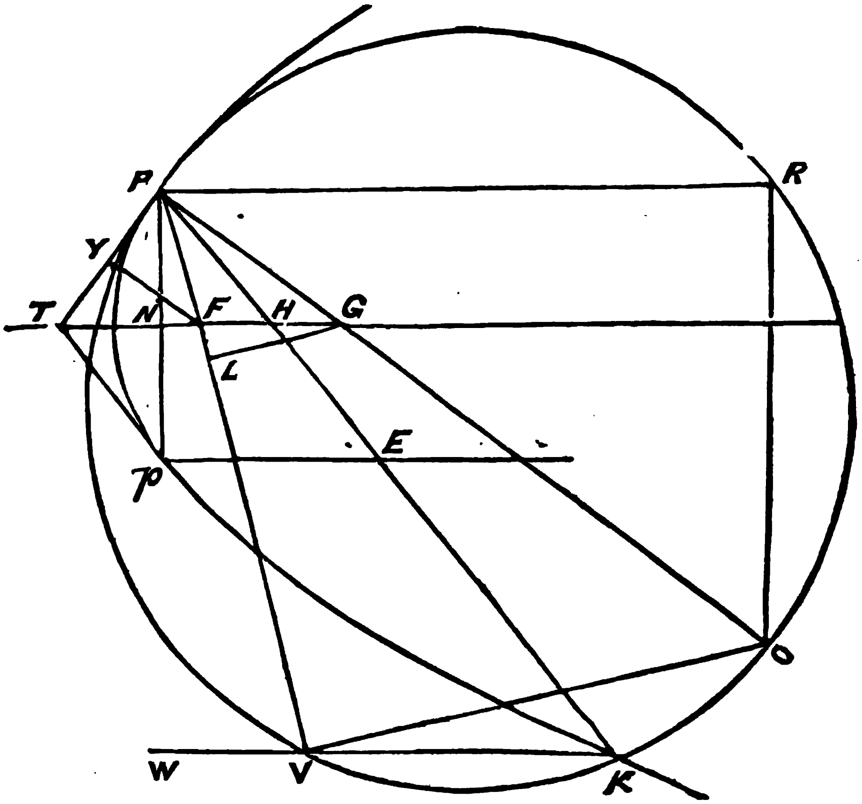
But by the similar triangles TGP, GPL, and since $TG = 2FP$

$$TG\ (2FP) : PG = PG : PL$$

Therefore $$2FP.PL = PG^2 \text{ and } 4FP.PL = 2PG^2$$

Therefore $$PO : PG = 2PG^2 : PL^2.$$

Fig. 88.



Hence we have $PV = PR = 4PF$

And $PO = \frac{4PF^2}{FY}$

Or $PO = \frac{2PG^2}{PL^2}$

Art. 160.

In the Ellipse and Hyperbola if the chord of intersection PK meet CD the diameter
FIGS. 89 & 90. conjugate to CP in L, then $PK.PL = 2CD^2$.

Draw the ordinate PNP′ to the axis and the tangent TPT′, and join P′T, CP′, the latter meeting PK in V, also let PK meet the axes in Z, Z′.

Then by parallels $PV : PZ' = VP' : P'C = TZ : TC$

Since P′K, P′T are parallel, as in the Parabola.

But by sim. tri. $TZ : TC = PZ : PL$

Therefore ex æquali $PV : PZ' = PZ : PL$

Therefore $PV.PL = PZ.PZ' = PT.PT' = CD^2$ (Arts. 100 and 137)

Hence $PK.PL = 2CD^2$.

COR.—If PQ be a chord of curvature in any direction and meeting CD in H, then

$$PQ.PH = 2CD^2$$

Let PO the diameter of curvature meet CD in E, then since the angles OKP, OQP are right angles, circles can be drawn through the points Q, H, E, O and K, L, E, O.

Therefore $OP.PE = KP.PL = QP.PH = 2CD^2$.

Art. 161.

FIGS. 89 & 90. Again, PG being the normal, let PFS be the chord of curvature through the focus) and draw GM perpendicular to PS, then GM is parallel to OS.

Therefore $PO : PG = PS : PM$

$= PS\,PM : PM^2$

But PS.PW, that is (by Arts. 95 & 119) $PS.AC = 2CD^2$ by the preceding Art.

∴ $PS : AC = 2CD^2 : AC^2$

By Arts. 102 & 138 $= 2PG^2 : BC^2$

And $PS.PM : AC.PM = 2PG^2 : BC^2$

But PM being equal to half the Latus Rectum $AC.PM = BC^2$

Therefore $PS.PM = 2PG^2$

And $PO : PG = 2PG^2 : PM^2$

We have therefore the following expressions for the diameter of curvature

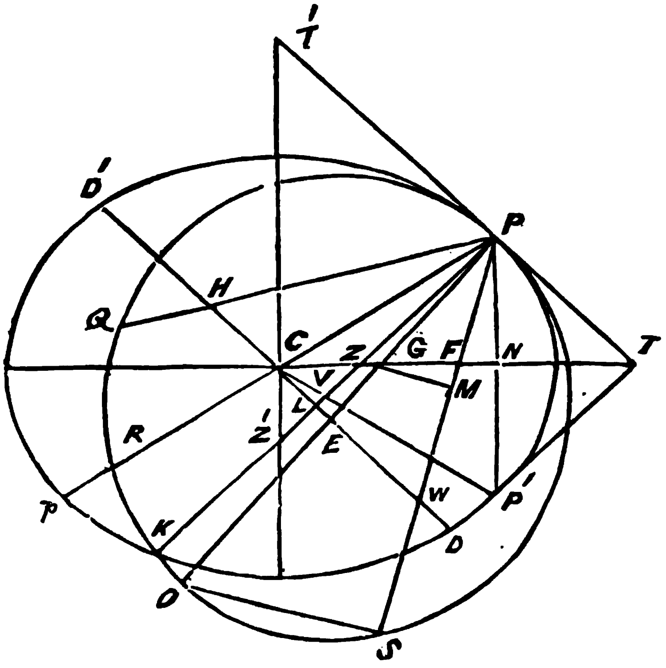
$$PO = \frac{2CD^2}{PE}$$

And

$$PO = \frac{2PG^2}{PM^2}.$$

Hence since this last expression is the same as that obtained for the Parabola, in any Conic the radius of curvature is to the normal as the square of the normal to the square of the semi Latus Rectum.

Fig. 89.



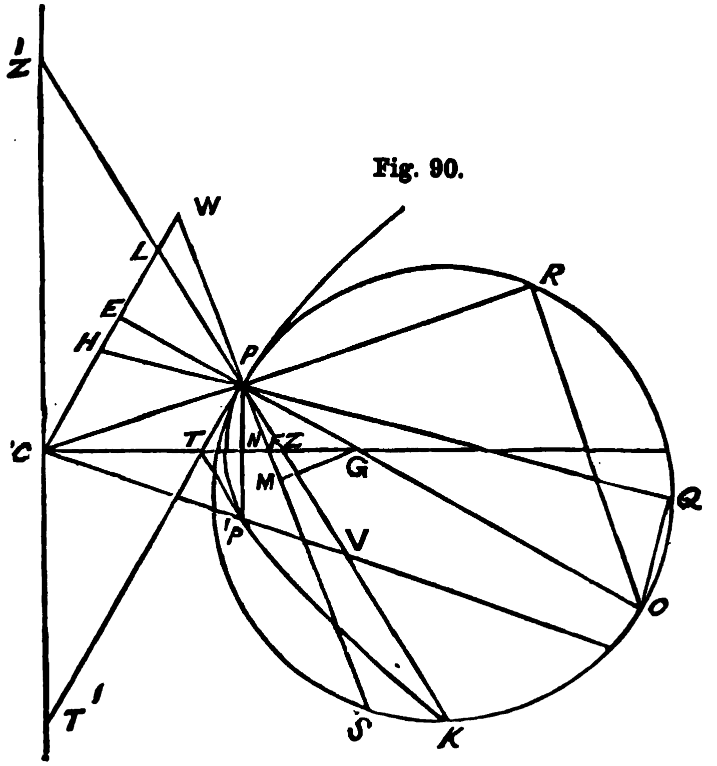


Fig. 90.

ANHARMONIC RATIO.

Art. 162. Def. 1.—Let A, B, C, D be four points in a straight line, then the ratio of AD.CB to AB.CD is called the Anharmonic Ratio of the range A, B, C, D, and is denoted by (ABCD.)

Def. 2.—If any four lines as VA, VB, VC, VD meet in a point they are called a pencil and the lines are called rays of the pencil; and the Anharmonic Ratio of a pencil is the Anharmonic ratio of the range in which its rays are intersected by any transversal this ratio being denoted by V.ABCD.

Let V1, V2, V3, V4 be any four lines meeting in the same point V, then if the four lines meeting in V be cut by any transversal whatever, the ratio of AD.CB to AB.CD is constant, where A, B, C, D are the points in which the transversal meets the lines V1, V2, V3, V4 respectively. For take an equal pencil of four lines, fig. (91) that is, one such that the lines of which it is formed contain equal angles with the lines of the pencil in (fig. 92), and draw NLK, QSR, NAM, PNQ parallel respectively to V′1, V′2, V′3, V′4, then it is evident by similar triangles that

$$MA : AN = NQ : NP = NK : KL = RS : SQ.$$

Now the transversal in (fig. 92) can only cut the pencil whose vertex is V in one of the following ways, viz., (1.4), (1.2′), (2.3′), (2.1′), (3.4′), (3.2′), (4.3′), (4′.1′). If therefore ECF, LDK, MAN, GBH be drawn parallel respectively to V1, V2, V3, V4; the ratio AD.BC to AB.CD may in every position of the transversal be expressed by one or other of the following ratios, viz., MA : AN or BH : BG or EC : CF or KD : DL. For example suppose the transversal to have the position (3.2′), that is abCd.

Then by sim. tri. $Cb : ba = CE : Va$

And $ad : dC = Va : CF$

$\therefore$ $ad.bC : ab.dC = CE : CF.$

And similarly it may be shown that when the transversal has the positions (1.4), (4′1′), (3.4′) $AD.BC : AB.CD = CE : CF.$

When it has the positions (2.3′), (2.1′) $AD.BC : DC.AB = BH : BG$

" " (1.2′) " " $= AM : AN$

" " (4.3′) " " $= DK : DL.$

Therefore in all cases AD.BC has to DC.AB one or other of the foregoing ratios, but these are all equal and constant from what has been shown from (fig. 91), therefore the ratio of AD.BC to AB.CD is constant however the transversal may be drawn.

There are also two other constant ratios deducible from the proceeding :—

For the sum of the rectangles AD.BC and AB.CD $=$ AB.BC $+$ BD.BC $+$ AB.CD $=$ AC.BC $+$ AC.CD $=$ AC.BD, therefore taking the transversal (32′)

$$AD.BC : AD.BC + DC.AB = CE : CE + CF$$

That is $AD.BC : AC.BD = CE : EF$

In like manner $AC.BD : DC.AB = EF : CF.$

And the same may be shown in all the other cases, therefore the ratios AD.BC : AC.BD and AC.BD : DC.AB are constant.

Fig. 91.

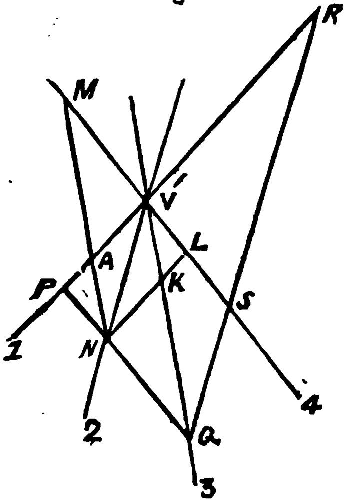
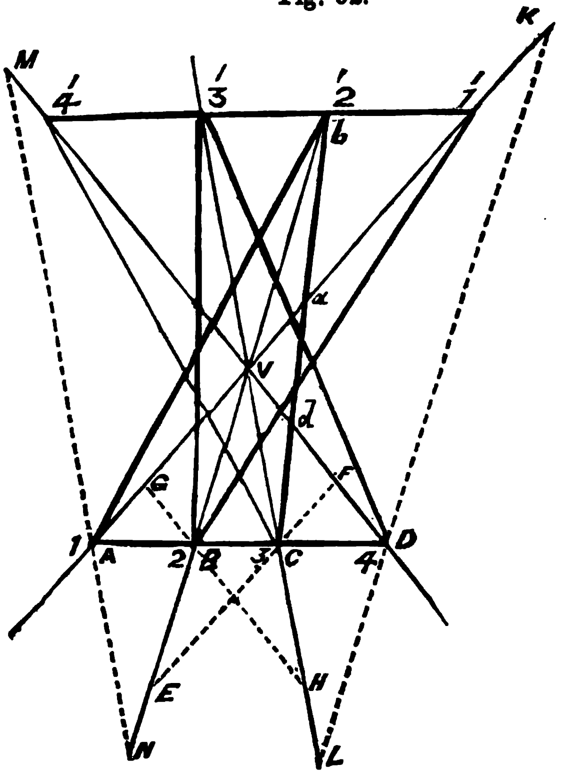


Fig. 92.



If three rays of a pencil be given and its Anharmonic ratio, the remaining ray may be found.

Let the three rays V'1, V'2, V'3 of the pencil in (fig. 91) and its Anharmonic Ratio be given, it is required to find the fourth ray of the pencil.

Take any point A in the ray V'1, and take AP so that V'A : AP in the given ratio, draw AN parallel to V'3 and meeting V'2 in N, join PN and draw V'4 parallel thereto, then V'4 is the fourth ray of the pencil required. For produce PN to meet V'3 in Q,

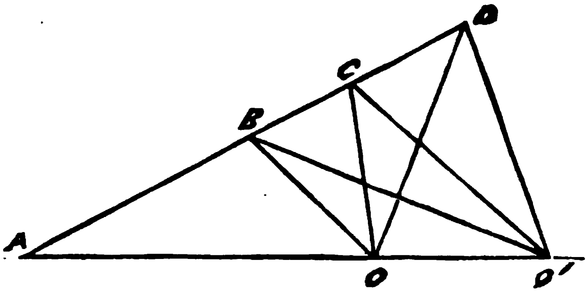
Then by parallels NQ : NP = AV : AP = a given ratio.

And from what has been proved, NQ : NP expresses the Anharmonic ratio of the pencil; therefore the ray V4 is determined so that the pencil of which it forms the fourth ray has a given Anharmonic Ratio.

Art. 163.

FIG. 93. **If two pencils have one ray in common and the same Anharmonic ratio, their remaining rays will intersect two and two in three points which lie in a straight line.**

Fig. 93.



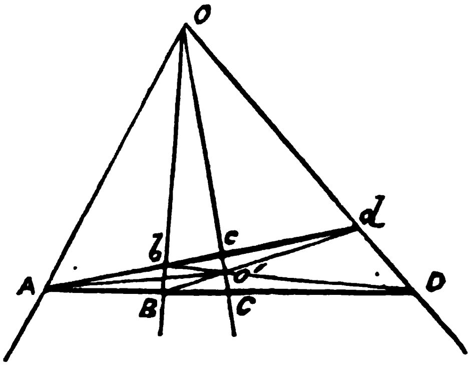
Let the pencils OA, OB, OC, OD; O′A, O′B, O′C, O′D have the ray OO′ common to both, then their other rays intersect in three points B, C, D which lie in one straight line. For join BC and produce it meet OO′ (produced) in A, then since the pencils have the same Anharmonic Ratio, it is evident that the line ABC produced must meet the fourth rays of each pencil in the same point, and therefore the three points B, C, D lie in one right line.

And if B, C, D be considered as three positions of a variable point, the condition that B, C, D should lie in the same straight line evidently is, that if B, C, D be joined with any two points O and O′, the joining lines should make with OO′ two pencils that have the same Anharmonic Ratio.

Art. 164

FIG 94. If ABCD, Abcd be two equal Anharmonic ranges, the lines Bb, Cc, Dd must meet in a point, and also the lines Bd, bD, Cc.

Fig. 94.



Produce the lines Bb, Cc to meet in the point O, and join AO, then since ABCD Abcd are equal Anharmonic ranges, it is evident that DO joined must pass through d. In the same manner if Bd, bD be joined meeting in O′, and AO′ be joined, it is equally evident, for the same reason, that CO′ must pass through c.

If Bb, Cc, Dd be considered as three positions of a variable straight line, the condition that this variable line should always pass through one point, is that it should cut any two transversals that meet one another in the same Anharmonic Ratio.

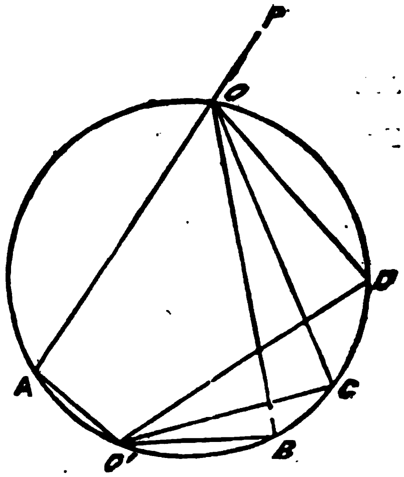
Art. 165.

Fig 95. If A, B, C, D be four points on the circumference of a circle, then if A, B, C, D be joined with any variable fifth point O, the Anharmonic Ratio of the pencil OA, OB, OC, OD is constant.

For while O remains between A and D all the angles at O remain unchanged in magnitude, and the truth of the proposition is evident. But if O should not lie between A and D, but between two of the fixed points as O′ between A and B for instance, then it is evident that BO′, CO′, DO′ make with AO′ the same angles that BO, CO, DO do with AO produced, (Eu. iii. 22) and therefore

$$O'(ABCD) = O(ABCD)$$

Fig. 95.



That is, the Anharmonic Ratio of four fixed points on the circumference of a circle from any fifth variable point is constant.

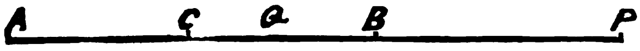
HARMONIC RATIO.

Art. 166.

Def.—A staight line is harmonically divided in two points when the whole line is to one of the extreme parts as the other extreme part is to the middle part; thus (fig. 96) the line AB is harmonically divided in P and Q if $AP : PB = AQ : QB$, or which is same proportion taken alternately, if $AP : AQ = BP : BQ$.

Fig 96.

Fig. 96.



If a straight line AB be bisected in C, and so divided in Q and P (P being in AB produced) that $PC.CQ = CB^2$, AB will be harmonically divided in P and Q.

For since $PC : CB = CB : CQ$

$\therefore$ $PC + CB : PC - CB = CB + CQ : CB - CQ$

That is $AP : BP = AQ : QB$

Therefore AB is harmonically divided in P and Q.

The condition that a range should be harmonic is evidently that its Anharmonic ratio should be one of equality, for if the anharmonic ratio of A, B, C, D (fig. 93) be one of equality $AD.BC = AB.CD$ or $AD : AB = DC : BC$, and therefore the range is harmonic. (Def.)

The condition that a pencil should be harmonic is of course that the range of any transversal by which it is cut should be harmonic, and this will always be the case when a parallel to one ray of a pencil is bisected by the middle one of the other three rays, thus in (fig. 92) if any of the lines MN, GH, EF, LK be bisected in the points A, B, C, D the pencil will be harmonic.

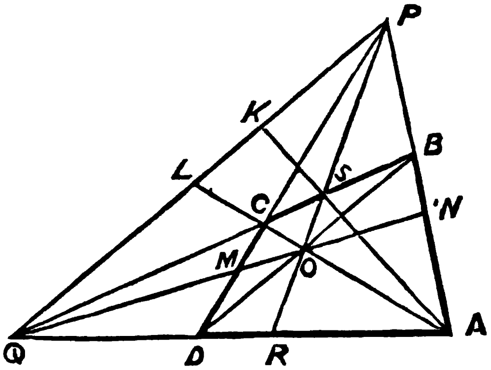
The internal and external bisectors of the angle formed by any two lines also form with the lines a harmonic pencil. For if (fig. 93) OB, OD be the internal and external bisectors of the angle AOC.

Then (by Eu. vi. 3 and A) $AB : BC = AO : OC = AD : DC$

Therefore AC is harmonically divided in B and D. (Def.)

HARMONIC PROPERTIES OF A COMPLETE QUADRILATERAL.

Fig. 97.



Art. 167.

Fig. 97. Let ABCD be a quadrilateral and produce its opposite sides to meet in P and Q, and draw the diagonals PQ, AC, BD, and complete the scheme, then every line of the figure is harmonically divided.

For consider the pencils QP, QC, QM, QD and OP, OC, OM, OD, these pencils evidently have the same anharmonic ratio, viz., that of PD.MC to PC.MD; but the anharmonic ratio of the pencil Q is also that of PA.BN to PB.NA, and that of O (the rays being taken in the same order) that of PB.NA : PA.BN; hence it follows that the ratio of PA.BN to PB.NA is one of equality, and therefore (by Art. 166) AB is harmonically divided in N and P, and therefore the pencils O and Q are harmonic, therefore CD is harmonically divided in M and P, and AD in R and Q; hence the pencils P and A are harmonic, and therefore every straight line in the figure being a transversal of one of the pencils A, O, P, Q is harmonically divided.

HARMONIC PROPERTIES OF THE CIRCLE.

Art. 168.

Fig. 98, & 99. If PT, Pt be a pair of tangents to a circle and Tt the chord of contact, then if the diameter PAB be drawn meeting Tt in Q, AB will be harmonically divided in P and Q; also if through P a parallel to Tt be drawn, then any straight line, drawn through either of the harmonic points P and Q to meet the parallel through the other, will be harmonically divided at the point and its intersections with the parallel and circle.

Let the centre of the circle be C, and join tC, then since PtC is a right angled triangle, and tQ is perpendicular to PC, therefore PC.CQ $=$ tC^2 $=$ CA^2; but CA $=$ CB, therefore AB is harmonically divided in P and Q (by Art. 166.)

Next let any line be drawn through P the exterior harmonic point to meet the circle in K and M and Tt in L, then KM is harmonically divided in L and P. Draw CV perpendicular to KM, therefore (by Eu. iii. 3) KV $=$ VM; also it is evident that a circle may be drawn through the five points P, t, C, V, T, and therefore (by Eu. iii. 36)

$$PL.LV = TL.Lt = KL.LM = KV^2 - VL^2.$$

Therefore PV.VL $=$ KV^2, and KM has been shown to be equal to VM, therefore KM is harmonically divided in L and P. (Art. 166.)

Lastly, let any line be drawn through the interior harmonic point Q and meeting the circle in K and M and PL parallel to Tt in L.

Draw CV perpendicular to KM, therefore KV $=$ VM (Eu. iii. 3); also circles may be described about the figures PtCT, PLCV, therefore (by Eu. iii. 36)

$$LQ.QV = PQ.QC = TQ.Qt = KQ.QM = KV^2 - VQ^2$$

Therefore LV.VQ $=$ KV^2, and therefore as in the last case KM is harmonically divided in P and Q.

Since (fig. 99), MK is harmonically divided in Q and L, it is evident that the chord of contact of a pair of tangents from L will pass through Q, and therefore conversely if chords be drawn through a fixed point Q in the diameter of a circle, pairs of tangents at their extremities intersect on a fixed straight line PL; hence the following definition :—

Def.—The locus of the points of intersection of tangents at the extremities of chords through a fixed point is called the POLAR of the point; also, if from points in a straight line pairs of tangents be drawn to a circle, the point in which all the chords of contact intersect is called the POLE of the line.

When the pole is without the circle, the polar is the chord of contact of tangents from the pole, and when the pole is on the curve the polar is the tangent at the point.

Fig. 98.

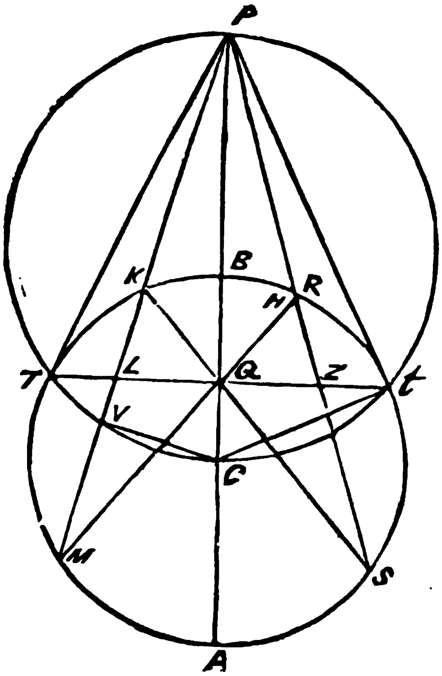
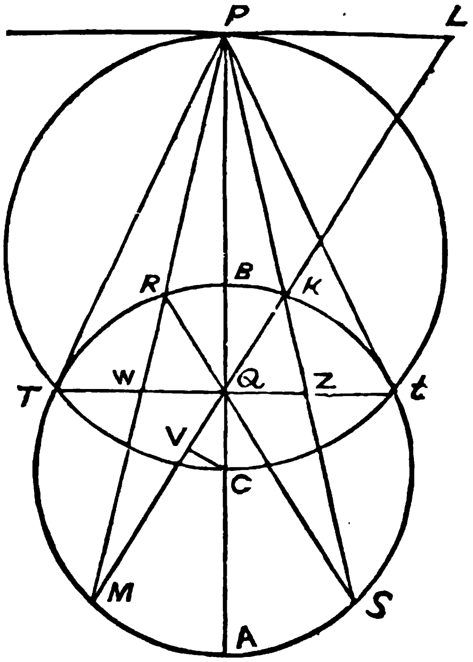


Fig. 99.



Note.—The above defiinition serves for the Conic sections as well as for the circle.

Art. 169.

Fig. 98. If through any point two straight lines be drawn cutting the circle the chords joining the alternate intersections with the circle meet on the polar of the point. For let KS, MR be drawn through the point Q (fig. 98), then RS, KM intersect on the polar of Q. Let KM meet the polar of Q in P, and let Tt be the chord of contact of tangents from P, therefore Tt passes through Q; but KM is harmonically divided in L and P (by Art. 168), and therefore QP, QK, QT, QM form an harmonic pencil. Join PS meeting Tt in Z and the circle again in H, therefore HS is harmonically divided in Z and P; but three of the points of division, viz., S, Z, P lie on the rays of the harmonic pencil just spoken of, therefore the fourth point H must lie in the remaining ray, that is, H and R coincide; hence KM, SR meet on the polar of Q.

Fig. 98. If the point P through which the lines PM, PS are drawn be without the circle, then it may be proved in a similar manner that KS, MR intersect on the polar of P; and it is also evident from Art. 164) that KR, MS meet on Tt produced.

ANHARMONIC PROPERTIES OF CONIC SECTIONS.

Art. 170.

If through any four points on a Conic, tangents be drawn, and also four other right lines meeting in any fifth point on the section, the anharmonic ratio of the four last lines is equal to that of the four points in which the four tangents are intersected by any fifth tangent to the curve.

Let there be a Conic of which F is the focus and XX′ a directrix, and let the tangents at four points A, B, C, D meet any fifth tangent at P in the points K, L, M, N. Join AP, BP, CP, DP, and produce them to meet the directrix in a, b, c, d respectively, and complete the figure.

Then by the Cor. of Prop. X, the angle KFL is either equal to aFb or its supplement, and the same is true of the angles LFM, bFc and MFN, cFd, consequently the pencils FK, FL, FM, FN and Fa, Fb, Fc, Fd have the same anharmonic ratio, that is, the anharmonic ratio of abcd, or what is the same thing of the pencil PA, PB, PC, PD, is equal to the anharmonic ratio of the four points K, L, M, N.

Cor. 1.—It is evident from what has just been shown that if A, B, C, D be any four fixed points on a Conic and P a variable point, that the anharmonic ratio P(ABCD) is constant.

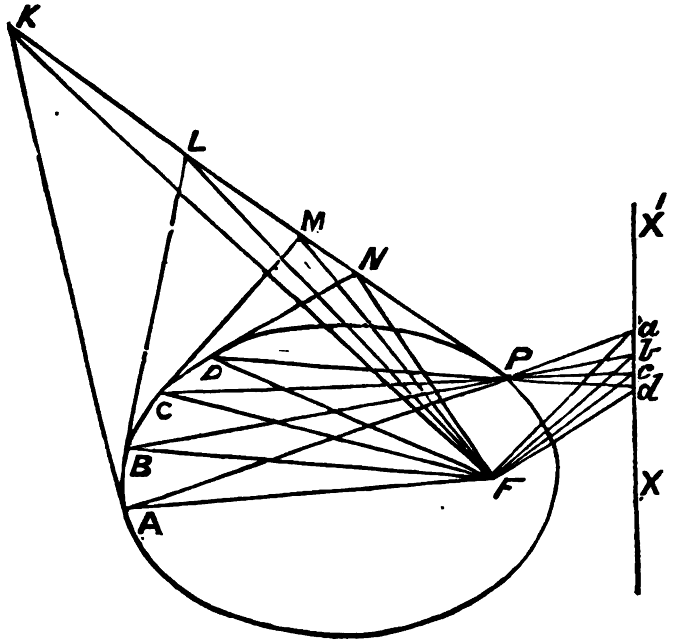
Cor. 2.—It is also evident that if the tangents at four fixed points A, B, C, D of a Conic meet a fifth variable tangent at P in the points K, L, M, N, that the anharmonic ratio of (KLMN) will be constant.

Art. 171.

One of the most distinguished of modern geometers, M. Chasles, has made the preceding theorem, which he demonstrates by the method of projections, the basis of his work on the Conic sections; he also demonstrates the reciprocals of the two corollaries just given in the early part of his work. As the reciprocal of the first of these corollaries may be demonstrated by means of the principle of the generating circle, of which it furnishes an illustration, will next be given, and afterwards a theorem (called by M. Chaslos that of Pappus) will be explained by means of tbe same principle.

Before proceeding to demonstrate the reciprocal of the first Cor. of the preceding theorem, it is necessary to observe that the Cor. may also be otherwise thus expressed: "If about two fixed points P and P′ on a Conic, two right lines which always intersect on the curve be made to revolve, then considering four successiye positions as A, B, C, D of the revolving lines, it is evident that P′A, PB, PC, PD and P′A, P′B, P′C, P′D form two homo-graphic pencils, that is, two pencils having the same anharmonic ratio."

Fig. 100.



The reciprocal then of the Cor. will be as follows: "If OP, OP_1, OP_2, OP_3 and O′P, $O'P_1$, $O'P_2$, $O'P_3$ be two homo-graphic pencils, the intersections P, P_1, P_2, P_3 of homologous rays lie on a Conic passing through O and O′.

In order then to demonstrate this:—

FIG. 101. Let O, O′ be two points on a Conic of which the focus is F and XX′ a directrix, and let OO′ meet XX′ in Z, and join FZ, and let S be the centre of a generating circle. Draw Sz parallel to OO′ and meeting the directrix in z, also draw zQQ′ parallel to FZ and meeting the circle in Q and Q′, then Q and Q′ are reciprocal points on the circle to O and O′ on the Conic, and if QS, Q′S be joined, they are respectively parallel to OF and O′F. (Art. 33.)

Let a line revolve about Q and successively assume any four positions Qpd, Qp_1e, Qp_2g, Qp_3h, and join Sd, Se, Sg, Sh, and through O draw DOP, EOP_1, GOP_2, HOP_3 respectively parallel to Sd, Se, Sg, Sh, and meeting the directrix in D, E, G, H, then if FP, FP_1, FP_2, FP_3 be drawn parallel respectively to SP, SP_1, SP_2, SP_3,* the points P, P_1, P_2, P_3 will be on the Conic. (By Art. 33.)

Again, let a straight line turn about Q′ and successively assume the positions Q′pk, $Q'p_1l$, $Q'p_2m$, $Q'p_3n$, and draw FK, FL, FM, FN respectively parallel to Q′K, Q′l, Q′m, Q′n. Join Sk, Sl, Sm, Sn, and draw lines through the points K, L, M, N respectively parallel to them, then these last four lines all pass through O and meet the curve again in the points P, P_1, P_2, P_3. This is evident from the principle of the generating circle; for take one of the parallels, viz., that to Sk through K, and let it meet FO′ in T, then by the principle of the generating circle, the point T is on the curve and therefore must coincide with O′; in the same way the parallels through L, M, N pass through O′ and the points P_1, P_4, P_3 respectively. Now by the circle (Art. 165) the anharmonic ratio of the pencil $Q(pp_1p_2p_3) = Q'(pp_1p_2p_3)$.

That is, S(degh) = S(klmn), but S(degh) = O(DEGH), and S(klmn) = O′(KLMN.

Therefore $O(DEGH) = O'(KLMN)$

That is $O(PP_1P_2P_3) = O'(PP_1P_2P_3)$

And P, P_1, P_2, P_3 have been shown to be points on the Conic, and they are the intersections of homologous rays of two homo-graphic pencils that have the same anharmonic ratio as the pencils $Q(pp_1p_2p_3)$ and $Q'(pp_1p_2p_3)$, which anharmonic ratio may be any whatever, therefore the intersections of homologous rays of two homographic pencils of which the vertices are at O and O′ are situated on the Conic upon which the points O and O′ are taken, hence the truth of the theorem.

---

* SP, SP_1, SP_2, SP_3 are omitted in the figure.

Fig. 101.

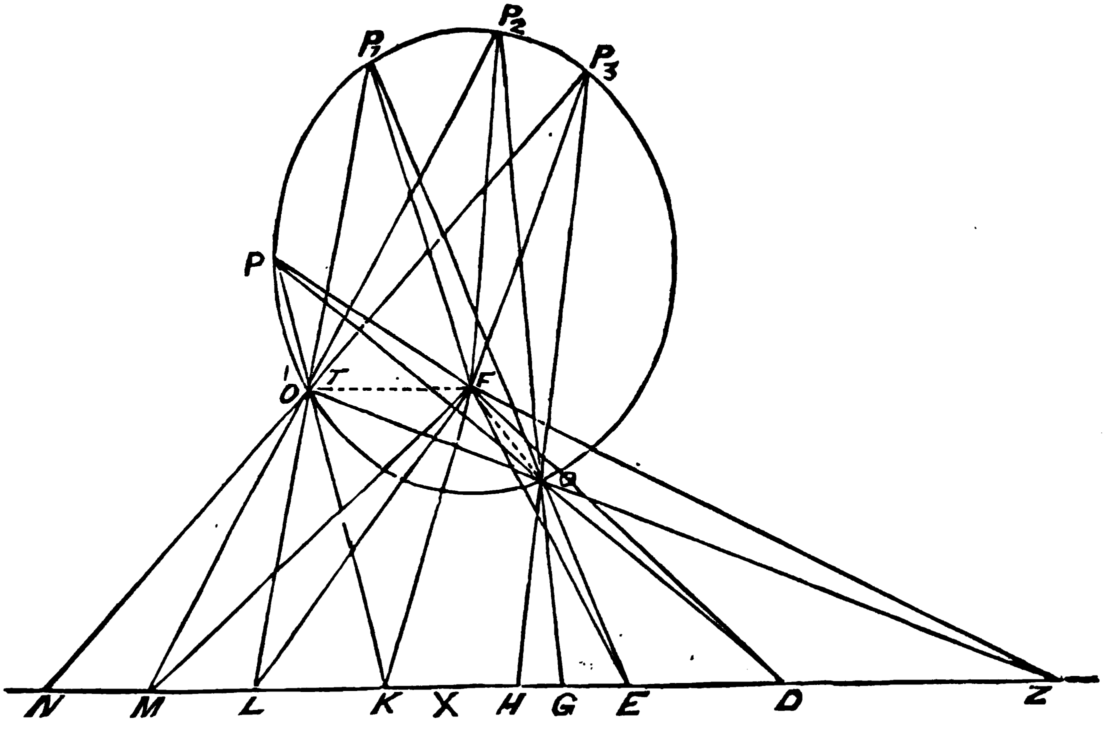
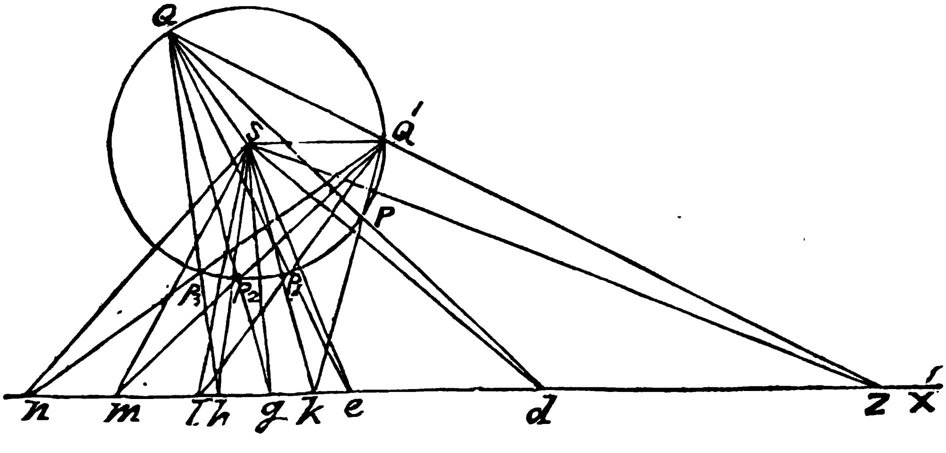


Fig. 101.



THEOREM OF PAPPUS.

Art. 172.

Fig. 102. If GIPQ be a quadrilateral inscribed in a Conic and M any fifth point on the curve, then the rectangle of the perpendiculars from M on the opposite sides GI, PQ shall have a constant ratio to the rectangle of the perpendiculars from the same point on the sides GQ, PI.

Let F be the focus and XX′ the directrix, produce the sides of the quadrilateral GIPQ to meet the directrix in the points K, V, L, W, and join FK, FV, FL, FW. Let S be the centre of a generating circle, and draw Sk, Sv, Sl, Sw respectively parallel to GI, IP, PQ, QI, then if through k, v, l, w parallels respectively to FK, FV, FL, FW be drawn, they will meet in four points A, B, C, D on the circle, that is the points A, B, C, D on the circle are reciprocal to the points G, I, P, Q on the Conic; in a similar manner find the point m on the circle reciprocal to M on the Conic, and complete the figure.

Putting then Z, Z′ and Y for the perpendiculars from the points A, C and S on the directrix.

$$\mathrm{ek} = \frac{\mathrm{eA.Ak\ sin\ eAk}}{Z} = \frac{\mathrm{eS.Sk\ sin\ eSk}}{Y}\ .$$

$$\mathrm{ew} = \frac{\mathrm{eA.Aw\ sin\ eAw}}{Z} = \frac{\mathrm{eS.Sw\ sin\ eSw}}{Y}$$

$$\mathrm{hl} = \frac{\mathrm{hC.Cl\ sin\ lCh}}{Z'} = \frac{\mathrm{hS.Sl\ sin\ hSl}}{Y}$$

$$\mathrm{hv} = \frac{\mathrm{hC.Cv\ sin\ hCv}}{Z'} = \frac{\mathrm{hS.Sv\ sin\ hSv}}{Y}$$

Therefore $$\frac{\mathrm{sin\ eSk\ .\ sin\ hSl}}{\mathrm{sin\ eSw\ .\ sin\ hSv}} = \frac{\mathrm{Ak.Sw\ .\ Cl.Sv}}{\mathrm{Aw.Sk\ .\ Cv.Sl}} \cdot \frac{\mathrm{sin\ eAk\ .\ sin\ lCh}}{\mathrm{sin\ eAw\ .\ sin\ vCh}}$$

Since eAk = vCh, and lCh and eAw are supplementary $$= \frac{\mathrm{Ak.Sw\ .\ Cl.Sv}}{\mathrm{Aw.Sk\ .\ Cv.Sl}}\ \text{a constant.}$$

But the angle eSk = MGI, hSl = MPQ, eSw = MGQ, and hSv = MPI by construction.

Therefore $$\frac{\mathrm{MG\ sin\ MGI\ .\ MP\ sin\ MPQ}}{\mathrm{MG\ sin\ MGQ\ .\ MP\ sin\ MPI}} = \text{a constant}$$

That is, if p_1, p_2, p_3, p_4 are the perpendiculars from M on GI, IP, PQ, QG respectively, and the above constant is represented by J

$$p_1p_3 = J \cdot p_2p_4$$

The value of the constant J is evidently (by sim. tri.) equal also to

$$\frac{\mathrm{FK\ .\ FL\ .\ GW\ .\ VP}}{\mathrm{FW\ .\ FV\ .\ GK\ .\ PL}}$$

Fig. 102.

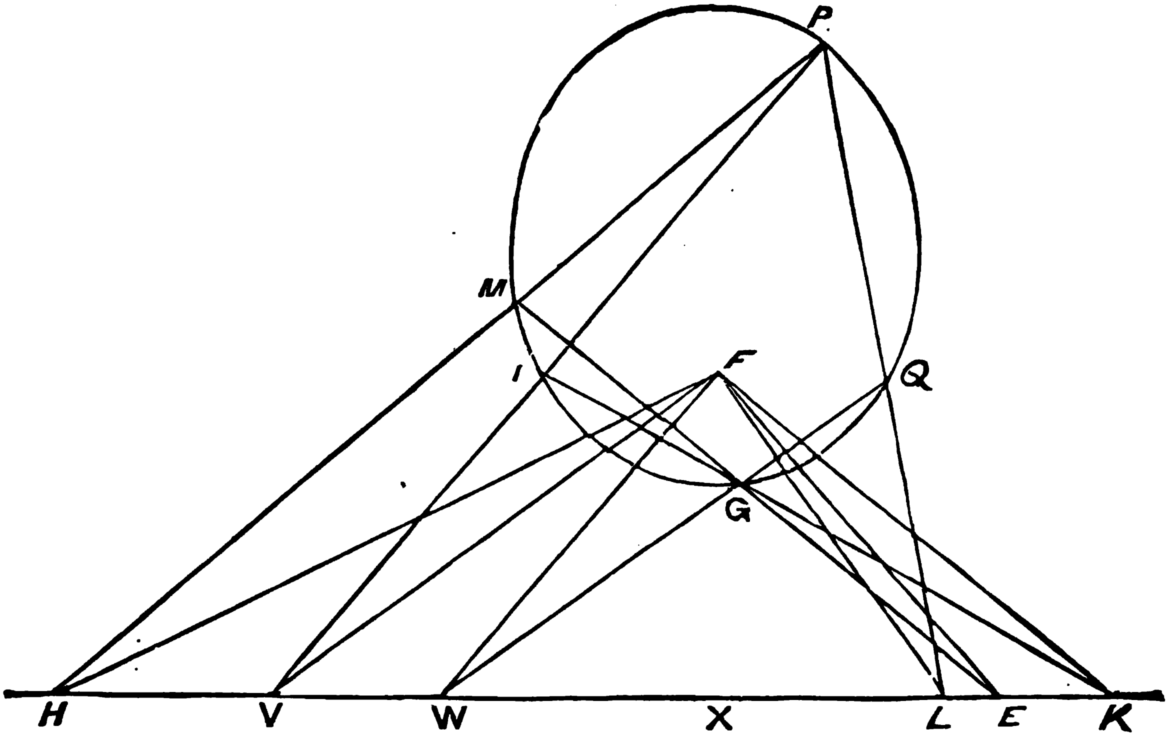
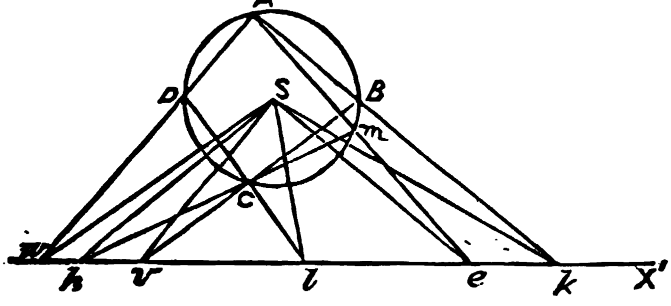


Fig. 102.



HARMONIC PROPERTIES OF THE CONIC SECTIONS.

Art. 173.

FIGS. 103 & 104. In the former part of the work, instances of the harmonic section of a straight line cutting a Conic section have occurred especially in the cases of the axes and diameters of the Ellipse and Hyperbola passing through the intersection of a pair of tangents, and the object of this Article is to show "that if through the intersection of a pair of tangents to any Conic section any straight line be drawn cutting the curve it will be harmonically divided by the curve and chord of contact.

Let AD, AE be a pair of tangents to a Conic, and AMLK any straight line drawn through A and cutting the curve in M and K and the chord of contact DE in L, then

$$AK : AM = KL : LM.$$

Through M and K draw parallels to DE cutting the curve again in m and k, and the tangents in H, h; G, g.

Then since the diameter through A bisects Mm, DE, Kk, and it also bisects Hh and Gg, hence $HM = mh$ and $GK = gk$.

And $GK : Kg = HM : Mh$

∴ $GK^2 : GK.Kg = HM^2 : HM.Mh$

Alt^{do} $GK^2 : HM^2 = GK.Kg : HM.Mh$

But by sim. tri. $GK^2 : HM^2 = KA^2 : AM^2$

And $GK.Kg : HM.Mh = KG.Gk : MH.Hm = GD^2 : DH^2$ (by Prop. VII)

$= KL^2 : LM^2$ by parallels

Therefore $KA^2 : AM^2 = KL^2 : LM^2$

And $KA : AM = KL : LM.$

COR. 1.—If ARr be the diameter passing through A and meeting DE in P, then if C be the centre, since Rr is divided harmonically in P and A and bisected in C, therefore (by Art. 166), $AC.CP = CR^2$.

COR. 2.—It is also evident from the preceding demonstration that if any tangent as AD meet a pair of parallel tangents as RV, rW in the points V and W, it will be harmonically divided, that is $WA : AV = WD : DV$.

NOTE.—In (fig. 104) if the dashed letters be taken, the demonstration of this theorem will be obtained for the Hyperbola in the case when the tangents A′D′, A′E′ touch opposite branches of the curve.

Fig. 103.

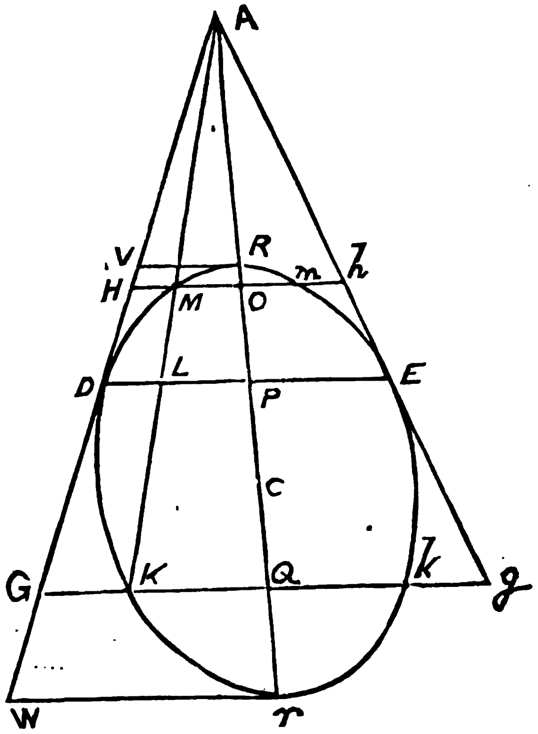
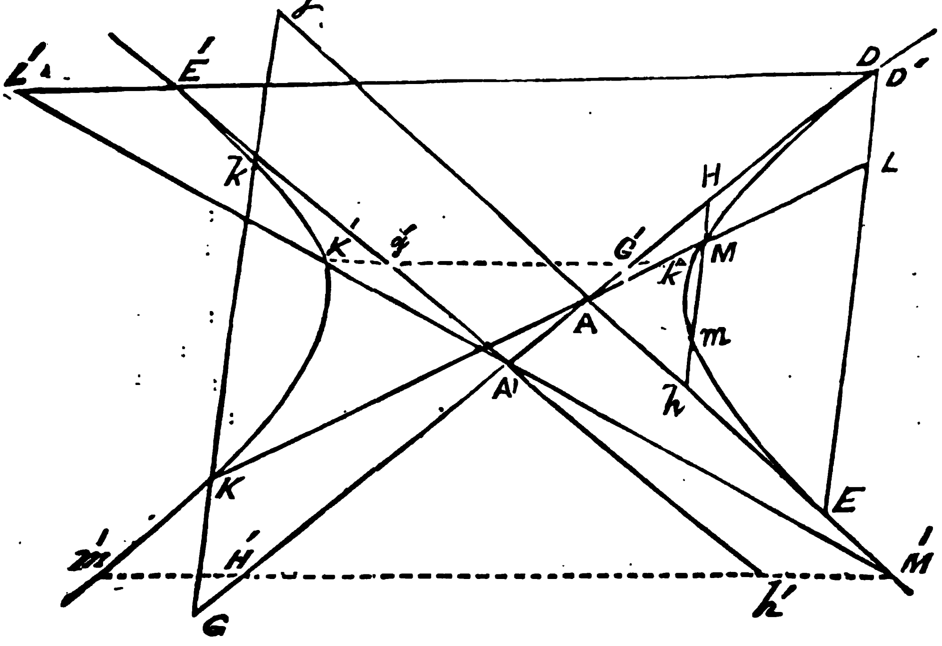


Fig. 104.



Art. 174.

Fig. 105. The preceding theorem is readily proved by means of the principle of the generating circle as follows: taking the case of the Ellipse for example. Let F be the focus (fig. 105) of the Ellipse, and S the centre of a generating circle pertaining to the Ellipse. Let SY, SY′ be the tangents to the Ellipse from the point S and YY′ the chord of contact, and SEKD any line drawn through S and cutting the Ellipse in D and E and YY′ in K, then DS : SE = DK : KE. From F draw the tangents FR, FR′ to the circle, and also join SR, SR′, FY, FY′, then FR. SY; FR′, SY′ meet on the directrix in two points V and V′, and SR is parallel to FY and SR′ to FY′ (by Art. 34.) Produce DS to meet the directrix in P, and join FP cutting the circle in d and e, and join FD, FE; Sd, Se, then by the principle of the generating circle FD is parallel to Sd and FE to Se. Produce YY′ to meet the directrix in W, and join FW, also produce RR′ to meet the directrix in Q, and join SQ, then (by Arts, 34 and 45) SF is at right-angles to FW, and by the circle it is at right-angles to RR′; therefore RR′ is parallel to FW, and conseqnently WYY′, SQ are parallel; therefore by sim. tri.

$$KP : PS = WP : PQ = FP : Pk$$

Therefore FK is parallel to Sk, and FD, FE have been shown to be parallel to Sd, Se; but SF, Se, Sk, Sd are a harmonic pencil by the circle, therefore FS, FE, FK, FD are also a harmonic pencil, and therefore DS : SE = DK : KE.

Art. 175.

Fig. 106. If two tangents be drawn to a Conic, any third tangent is divided harmonically by the two tangents, the curve, and the chord of contact.

Let AP, AQ be a pair of tangents to a Conic, and PQ their chord of contact, and let any third tangent at R meet AP, AQ in D and E and PQ in G, then

$$EG : GD = ER : RD.$$

Through E draw ELMN parallel to AP and meeting the curve in L and N and PQ in M.

Then by Prop. VII	$NE.EL : EQ^2 = AP^2 : AQ^2$
By sim. tri.	$= EM^2 : EQ^2$
Therefore	$NE.EL = EM^2$
By sim. tri.	$EG^2 : GD^2 = EM^2 : DP^2$
	$= NE.EL : DP^2$
By Prop. VII	$= RE^2 : RD^2$
Therefore	$EG : GD = RE : RD$

Therefore the tangent at R is harmonically divided.

Cor.—If AR be joined and produced to meet PQ in S, then PQ will be harmonically divided in S and G; for AG, AP, AR, AQ is a harmonic pencil and therefore PQ is harmonically divided in S and G.

Fig. 105.

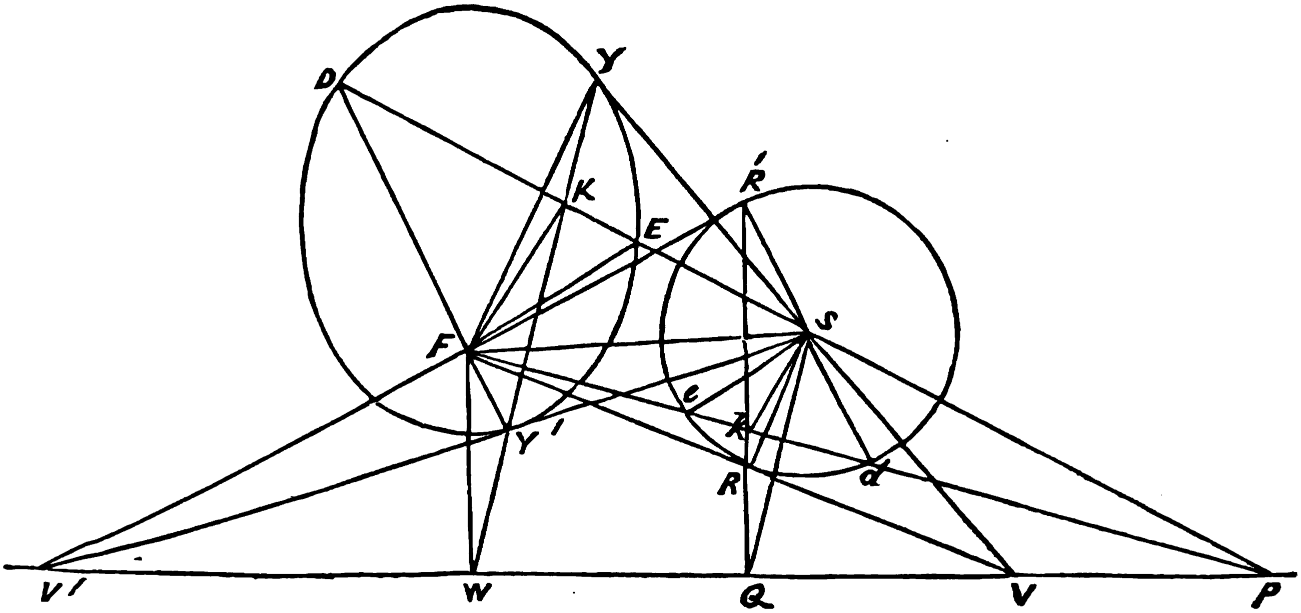
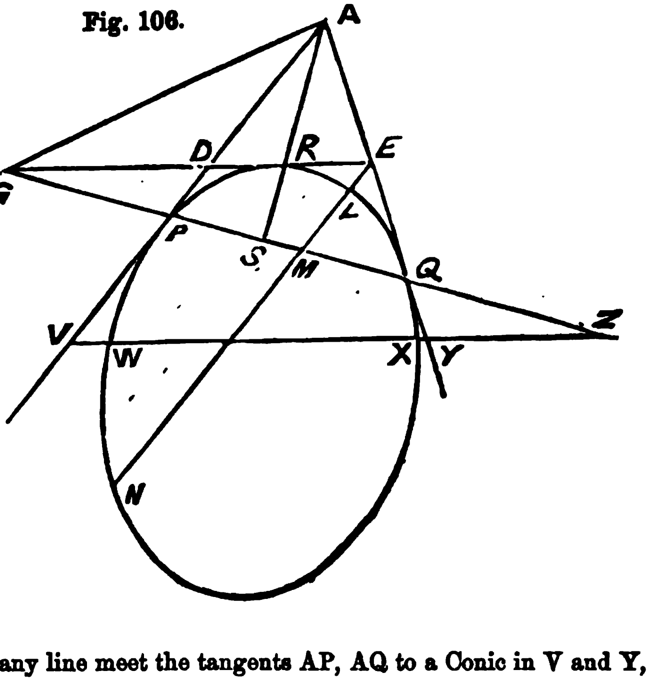


Fig. 106.



Art. 176.

Fig. 106. Again, let any line meet the tangents AP, AQ to a Conic in V and Y, the curve in W and X, and the chord of contact PQ in Z.

Then $VW.VX : VZ^2 = XY.YW : YZ^4$

For let the tangent at R be parallel to VZ.

Then (by Prop. VII) $VW.VX : DR^2 = VP^2 : DP^2$

By sim. tri. $= VZ^2 : GD^2$

And (by Prop. VII) $XY.YW : RE^2 = QY^2 : EQ^2$

By sim. tri. $= YZ^2 : EG^2$

But by Art. 175 $DR : RE = GD : EG$

Therefore $VW.VX : VZ^2 = XY.YW : YZ^2$.

Art. 177.

Figs. 107 & 108. **Tangents at the extremities of chords of a Conic sections passing through a fixed point meet on the polar* of the point.**

First let the fixed point be within the section as B, and let AD be the polar of B (figs. 107 and 108), let any line SBT be drawn through B meeting the section in S, T, the tangents at S and T meet on AD; for let them meet in the point M, and join AM, draw the diameter CM meeting ST in N, also draw the tangent PL and KP an ordinate to CM, then since $MC.CN = LC.CK$

$\therefore$ $MC : CL = CK : CN$

By sim. tri. $= CP : CB$

Since $AC.CB = CP^2$ (Arts. 94 & 118) $= AC : CP$

Therefore AM is parallel to PL, but PL is parallel to the ordinates of the diameter AC, that is to AD; hence AMD is a straight line, and it is the polar of B, therefore the tangents at S and T meet on the polar of B.

Figs. 109 & 110. Next let the fixed point be without the section as A (figs. 109 and 110), and let QBR be the polar of A, and let any line AST be drawn through A to meet the section in S, T; the tangents at S and T meet on QBR. For making a similar construction to the previous case $MC.CN = LC.CK$

$\therefore$ $MC : CL = CK : CN$

By sim. tri. $= CP : CA$

Since $AC.CB = CP^2$ by def. $= CB : CP.$

Therefore BM is parallel to PL, but PL being a tangent at P is parallel fo the ordinates of the diameter AC, therefore BM is the polar of A, wherefore the tangents at the extremities of the chord AST meet on the polar of A.

Note.—When the Conic is a Parabola, the lines NM, KL are bisected by the curve, and for the rest the proof is the same; also when the chord ST coincides with the diameter AC, the tangents SM, TM coincide with the tangents at the extremities of the diameter AP, and consequently are parallel to the chord of contact QR.

Cor.—Hence it is evident that 'if chords of a Conic be drawn through a fixed point pairs of tangents at their extremities will intersect on a fixed line, viz., on the polar of the point through which the chords are drawn,' and conversely 'if pairs of tangents be drawn from points in a fixed straight line, the chords of contact will all intersect in one point, viz., in the pole of the fixed line.'

---

*To avoid repetition the reader is referred to the definition of the terms Pole and Polar given on (page 126), as that definition serves for the Conic sections equally with the circle.

Fig. 107.

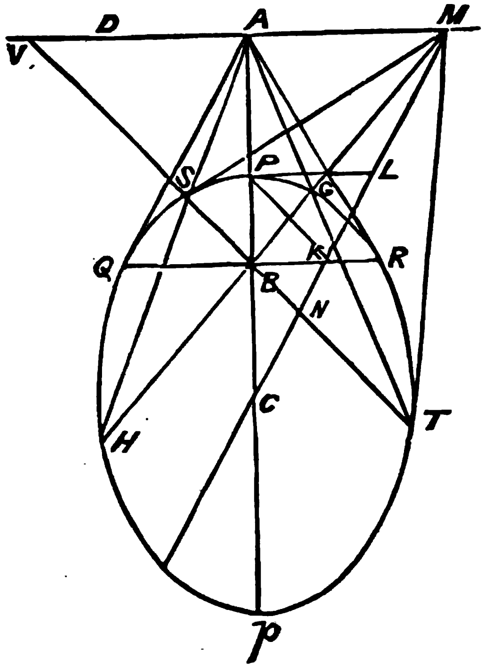


Fig. 108.

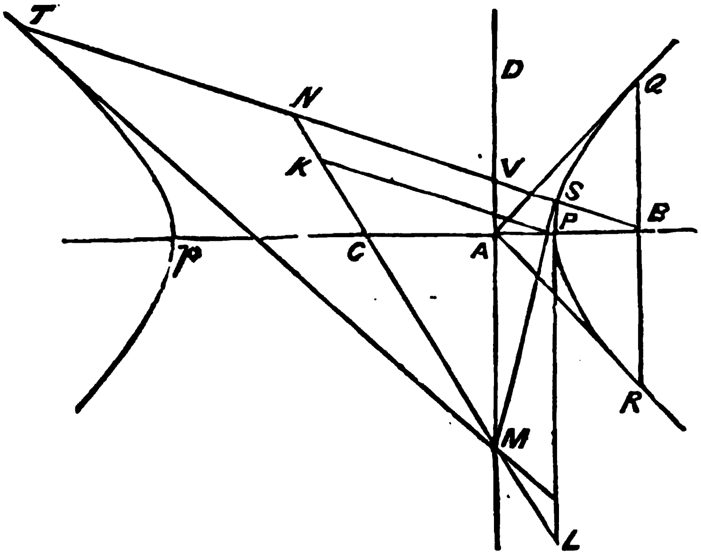


Fig. 109.

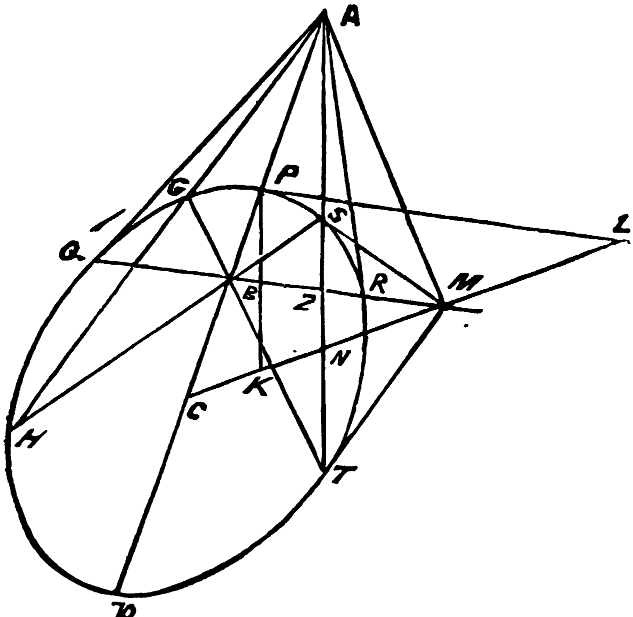
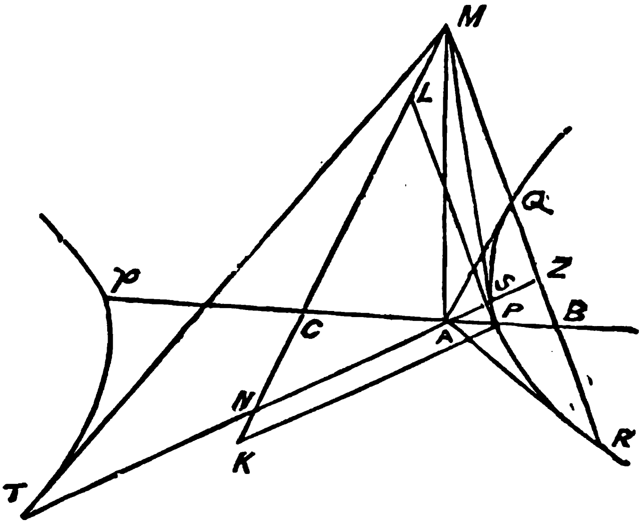


Fig. 110.



Art. 178.

A straight line drawn through any point is divided harmonically by the point, the curve, and the polar of the point.

When the point through which the line is drawn is without this Conic, this has been already proved in Art. 174, and when the point is within the Conic as B (figs. 107), then supposing ST to be the line drawn through B, produce ST to meet the polar of B, viz., AD in V; therefore by what has just been proved (Art. 177), the chord of contact of tangents from V passes through B, and consequently ST is harmonically divided in B and V.

Art. 179.

The polars of two points intersect in the pole of the line joining the points.

This is evident from what has been shown, for if A and M be the points (figs. 107 and 108) their polars QR and ST intersect in B the pole of AM; and similarly (in figs. 109 and 110) the point Z the intersection of QR and ST is the pole of AM.

Art. 180.

It may also be shown in precisely the same way as was done in the circle, that if through any point two lines be drawn cutting a Conic section the straight lines joining their alternate intersections with the Conic meet on the polar of the point. (See Art. 169.)

EXAMPLES.—CURVATURE.

1.—The radius of curvature at the extremity of the Latus Rectum of a Parabola is equal to twice the normal.

2.—The radius of curvature at any point of a Conic is to the normal at that point in the duplicate ratio of the normal to the semi Latus Rectum.

3.—The circle of curvature at P in a Parabola, cuts off from the diameter through P a portion equal to the parameter of that diameter.

4.—The circle of curvature at the end of the Latus Rectum intersects the Parabola on the normal at that point.

5.—P and p are points on a Parabola on the same side of the axis, PN and pn are perpendiculars on the axis, the normals P and p meet at a point Q, show that the distance of Q from the axis is $= \dfrac{2PN \ : \ pn\,(PN + pn)}{(\text{Latus Rectum})^2}$

6.—The circle of curvature of a point on an ellipse equidistant from the axis minor and one of the directrices passes through one of the foci.

7.—In the Parabola 2FP is a mean proportional between FA and the portion of the axis cut off by the circle of curvature at P.

8.—The circles of curvature at the extremities P, D of two conjugate diameters of an Ellipse meet the Ellipse again in Q, R respectively, show that PR is parallel to DQ.

9.—In the rectangular Hyperbola, the radius of curvature at a point P varies as CP^3

10.—The circle of curvature at the extremity of one of the equi-conjugate diameters of an Ellipse passes through the other extremity of that diameter.

11.—There are three points P, Q, R on an Ellipse whose circles of curvature pass through a given point on the curve, these lie on a circle passing through the point and form a triangle of which the centre of the Ellipse is the intersection of the bisectors of the sides.

12.—If a tangent be drawn from any point of a Parabola to the circle of curvature at the vertex, the length of the tangent will be equal to the abscissa of the point measured along the axis.

13.—The central chord of curvature at any point of the rectangular Hyperbola is equal to the diameter of the curve which passes through the point.

14.—The radius of curvature at any point P of a rectangular Hyperbola is to CP in the duplicate ratio of CP to CA.

15.—Find the points at which the radius of curvature is a mean proportional between the semi-axes of an Ellipse.

16.—Prove that there is a point P on an Ellipse such that if the normal at P meet the Ellipse in Q, PQ is a chord of the circle of curvature at P, and finds its position.

17.—The chord of curvature at a point P of a rectangular Hyperbola, perpendicular to an Asymptote, is to CD as CD to 2PN where PN is the distance of P from the Asymptote.

18.—The foci of an Ellipse, one extremity of the axis minor, and the centre of the osculating circle at that extremity, lie on a circle.

19.—In an Ellipse the circle of curvature cannot pass through the focus if the semi-axis major is greater than the distance between the foci.

20.—The sum of the focal chord of curvature at any point on an Ellipse, and the focal chord of the Ellipse parallel to the diameter through the point is constant.

HARMONIC PROPERTIES.

21.—OA, OB, OC being three lines given in position, show that there are three other straight, each of which forms with OA, OB, OC an harm^{o}nic pencil; and that each of the three OA, OB, OC forms with the second three a harmonic pencil.

22.—CA, CB are two tangents to a Conic section, O a fixed point in AB, POQ any chord of the Conic, prove that the intersections of AP, BQ, and also of AQ, BP lie on a fixed straight line which forms with CA, CO, CB an harmonic pencil.

23.—If OT, OT′ are tangents to a Conic, OPQ any line cutting the Conic in P, Q, then if R be the pole of OPQ, OT, OPQ, OT′ and OR form an harmonic pencil.

24.—Show that the Asymptotes of an Hyperbola form with any pair of conjugate diameters an harmonic pencil.

25.—Show that the centre of a Conic is the pole of a line at an infinite distance.

26.—If from a given point in the axis of a Conic a chord be drawn, the perpendicular from the pole of the chord upon the chord will meet the axis in a fixed point.

27.—If two Conics have the same focus and directrix, and a focal chord be drawn, the four tangents at the points where it meets the Conics intersect in the same point on the directrix.

28.—TP, TQ are tangents to an Ellipse, and PQ meets the directrices in Z, Z′, show that PZ QZ′ subtend equal angles at T.

CONES.

29.—From a given cone it is required to cut an Ellipse whose eccentricity is in the ratio of one to two.

30.—Find the least angle of a cone from which it is possible to cut an Hyperbola, whose eccentricity shall be ratio of two to one.

31.—Show that all sections of a right cone made by planes parallel to a tangent plane of the cone are Parabolas, and that the foci lie on a cone having with the first a common vertex and axis.

32.—Prove that all sections of a cone by parallel planes are Conics having the same eccentricity, and find the locus of the foci.

33.—If two plane sections of a right cone be taken having the same directrix, the foci corresponding to that directrix lie on a straight line which passes through the vertex.

34.—Given a right cone and a point within it, there are but two sections which have this point for focus, and the planes of these sections make equal angles with the straight line joining the given point and the vertex of the Cone.

35.—The Latus Rectum of a Parabola cut from a given cone varies as the distance between the vertices of the cone and Parabola.

36 —The foci of all elliptical sections of the same eccentricity lie on two cones.

37.—Show how to cut a right cone so that the section may be an Ellipse whose axes are of given lengths.

38.—Show how to cut from a given cone an Hyperbola whose Asymptotes shall contain the greatest possible angle.

39.—The angle between the Asymptotes of a section being given, determine the locus of either focus.

40.—A plane through the vertex parallel to the plane of an Hyperbolic section intersects the cone in generating lines which are parallel to the Asymptotes of the section.

41.—If two cones be described touching the same two spheres, the eccentricities of the two sections of them made by the same plane bear to one another a ratio constant for all positions of the plane.

42.—One vertex of any right circular cone which contains a given Ellipse will be on a certain Hyperbola, and the axis of the cone will be a tangent to the Hyperbola.

43.—If the vertical angle of a cone, vertex V, be a right angle, P any point of a Parabolic section and PN perpendicular to the axis of the Parabola, $VP = 2AF + AN$, A being the vertex and F the focus of the curve.

44.—In the above questions from Nos. 29 to 43, wherever the words "right cone" are used, substitute the words "obliqe cone," and ascertain if the same results hold good, that is, resolve the same problems in the oblique cone as for the right.

MISCELLANEOUS.

45.—If from the same point P in an Ellipse two focal chords PFp, PF′q be drawn, and L be the Latus Rectum of the axis, prove that $Pp.Pq : 4PF.Fp = 4PF'F'q : L^2$.

46.—Given two fixed points of a Conic and the directrices, find the locus of the focus.

47.—A line cuts two concentric and similary situated Ellipses in P, Q, q, p, if the line move parallel to itself, PQ.Qp is constant.

48.—If a circle be described passing through any point P of a given Hyperbola, and the extremities of the transverse axis, and the ordinate NP be produced to meet the circle in Q, the locus of Q is an Hyperbola.

49.—If from a point T in the director circle of an Ellipse, tangents TP, TP′ be drawn, the line joining T with the intersection of the normals at P and P′ passes through the centre.

50.—OP, OQ touch a Parabola at the points P, Q; another line touches the Parabola in R, and meets OP, OQ in S, T; if V be the intersection of the lines PT, SQ; O, R, V are in a straight line.

51.—The diameter through a point P of a Parabola meets the tangents at the vertex in Z, the normal at P and the focal distance of Z will intersect in a point at the same distance from the tangent at the vertex as P.

52.—If PFp be a focal chord of a Conic, Q any point of the Conic, and if PQ, pQ meet the directrix in D and E, DFE is a right-angle.

53.—If TP, TQ be two tangents to a Parabola, such that the chord PQ is normal at P, then $PQ : PT = PN : AN$; PN and AN being ordinate and abscissa of P.

54.—PFp is a focal chord of a Parabola, RD is the directrix meeting the axis in D, Q is any point on the curve, prove that if QP, Qp produced meet the directrix in R, r, half the Latus Rectum will be a mean proportional between DR, Dr.

55.—PQ is any chord of a Parabola cutting the axis in L; R, R′ are the two points in the Parabola at which this chord subtends a right angle; if R, R′ be joined meeting the axis in L′, LL′ will be equal to the Latus Rectum.

56.—Prove that the Conic which touches the sides of a triangle and has its centre at the centre of the nine point circle, has one focus at the ortho-centre and the other at the centre of the circumscribing circle.

57.—From Q the middle point of a chord PP′ of an Ellipse whose focus is F, QG is drawn perpendicular to PP′ to meet the major axis in G, prove that

$$2.FG : FP + FP' = SA : AX.$$

58.—Tangents at the extremities of a pair of conjugate diameters of an Ellipse meet in T, prove that FT, F′T meet the conjugate diameters in four points which lie on a circle.

59.—The angle between the focal distance and tangent at any point of an Ellipse is half the angle subtended at the focus by the diameter through the point.

60.—Through a given point within a Parabola draw a chord which shall be divided in a given ratio in that point.

61.—If a chord RPQV meet the directrices of an Ellipse in R and V, and the circumference in P and Q, then RP and QV subtend, each at the focus nearer to it, angles of which the sum is equal to the angle between the tangents at P and Q.

62.—A sphere is described about the vertex of a right cone as centre, the latera recta of all sections made by tangent planes to the sphere are equal.

63.—A circle of curvature of an Ellipse at P passes through the focus F, FM is drawn parallel to the tangent at P to meet the diameter of the Ellipse in M, show that it divides this diameter in the ratio 3 : 1.

64.—Two focal chords (PFp, QFq) of a central Conic at right angles to each other are produced to meet the nearer directrix in R, r, and central radii CP′, CQ′ are drawn parallel to them, show that if BCB′ be the minor axis $\frac{CP'^2}{RP.Rp} + \frac{CQ'^2}{rQ.rq} = \frac{CS^2}{CB^2}$

65.—Prove that the tangents PT, QT from any two points P and Q of a Parabola, as mutually limited by each other, are as their distances from the focus.

66.—To a given Ellipse to draw a tangent terminated by the major axis produced and a given ordinate to the same axis, such the parts between the point of contact and the produced ordinate and axis may have to one another a given ratio.

67.—If from any point P in a circle concentric with an Ellipse, and the radius of which is equal to the distance between the ends of the major and minor axes, a pair of tangents be drawn to the Ellipse and produced to meet the circle in the points S and S′, prove that the line SS′ is parallel to the polar of P.

68.—Given a pair of semi-conjugate diameters of an Ellipse, to find any number of points on the curve.

69.—A circle is described through the foci of a Conic, and from a point P on it tangents PA, PB are drawn to the curve meeting the circle again in Q and R, show that the other tangents to the Conic from Q and R meet on the circle.

70.—Given the focus, directrix, and length between the extremities of the major and minor axes of an Ellipse, show how to find the major axis and the directions of conjugate diameters geometrically.

71.—The straight line joining the foci of a Conic subtends at the pole of any chord half the sum or difference of the angles which it subtends at the extremities of the chord.

72.—If O be that point in the normal to a Parabola at P, through which if any chord pass it will subtend a right angle at P, PO will be bisected by the axis.

73.—If two points be fixed in the axis of a Parabola at equal distances from the focus, prove that the difference of the squares on the perpendiculars drawn from them to any tangent is constant.

74.—If PQ be a chord of an Ellipse meeting the circle of curvature at P in Q′, prove that $PQ' : PQ = d^2 : d'^2$ where d and d′ are the diameters parallel to the tangent at P and to the chord respectively.

75.—ACB is the diameter of a circle, centre C; P is a Parabola, focus C, vertex A; H is an Hyperbola with one asymptote parallel to ACB, with one focus at C, and touching P and the circle, show that the transverse axis of H is equal to the radius of the circle.

76.—In the right circular cone if C, C′ be the centres of a pair of focal spheres, show that the sphere on C, C′ contains the auxiliary circle of the section.

77.—Prove that the polar of a point with respect to a Conic meets the directrices on the tangents from that point to a confocal.

78.—If a circle roll on the inside of the circumference of a circle of double its radius, any point in the area of the rolling circle traces out an Ellipse.

79.—A given straight line has its end moveable on two straight lines at right angles to each other, the path of any given point in the moving line is an Ellipse.

80.—The normal at a point P of an Ellipse meets the curve again in P′, and through O the centre of curvature at P the chord QOQ′ is drawn at right angles to PP′, prove that

$$QO.OQ' : PO.OP' = 2PO : PP'.$$

81.—A focal chord PFQ is drawn to a Conic of which C is the centre; the tangents and normals at P and Q intersect in T and K respectively, show that FP, FP, FK, FC form a harmonic pencil.

82.—Prove the following construction for a pair of tangents from any external point T to an Ellipse of which the centre is C:—Join CT, let TPCP′T a similar and similarly situated Ellipse be drawn, of which CT is the diameter, and P, P′ its points of intersection with the given Ellipse; TP, TP′ will be tangents to the given Ellipse.

83.—Given a geometrical construction by which a cone may be cut so that the section may be an Ellipse or Hyperbola of given eccentricity.

84.—If the curve formed by the intersection of any plane with a cone be projected upon a plane perpendicular to the axis, prove that the curve of projection will be a Conic section having its focus at the point in which the axis meets the plane of projection.

85.—Find the locus of the points of contact of tangents to a series of confocal Ellipses from a fixed point in the axis major.

86.—An Ellipse slides between two lines at right angles to each other, find the locus of its centre.

87.—OQ, OQ′ are tangents to an Ellipse from an external point O, and OR is a diagonal of the parallelogram of which OQ, OQ′ are adjacent sides, prove that if R be on the Ellipse, O will lie on a similar and similarly situated concentric Ellipse.

88.—An Ellipse touches the sides of a triangle, prove that if one of its foci move along the arc of a circle passing through two of the angular points of the triangle, the other will move along the arc of a circle through the same two angular points.

89.—If a given point be the focus of any Hyperbola passing through a given point and touching a given straight line, prove that the locus of the other focus is an arc of a fixed Hyperbola.

90.—Show that the locus of the point of intersection of tangents to a Parabola making with each other a constant angle equal to half a right angle is an Hyperbola.

91.—PCP′ is a given diameter of an Ellipse, QVQ′ an ordinate to PP′; the locus of the intersection of PQ′, P′Q is an Hyperbola.

92.—Two sides of a quadrilateral are given in magnitude and position, the quadrilateral be such that a circle can be inscribed in it, the locus of the point of intersection of the other two sides is an Hyperbola.

93.—An Hyperbola having a given transverse axis, has one focus fixed, and always touches a fixed straight line, the locus of the other focus is a circle.

94.—A given straight line has its ends moveable on two straight lines at right angles to each other, the path of any given point in the moving line is an Ellipse.

95.—The locus of the intersections of the focal distances of any point on an Ellipse with the central perpendicular upon the tangent at the same point consists of two circles.

96.—The locus of the foci of all Parabolas touching the three sides of a triangle is a circle, find its centre geometrically.

97.—A focus, tangent, and the eccentricity of a Conic section being given, show that the locus of its centre is a circle.

98.—In an Ellipse if a line be drawn through the focus making a constant angle with the tangent, prove that the locus of the point of intersection with the tangent is a circle.

99.—If an Hyperbola be described touching the four sides of a quadrilateral inscribed in a circle, and one focus lie on a circle, the other focus will also lie on the same circle.

100.—Given in an Ellipse a focus and two points, the locus of the other focus is an Hyperbola.

101.—If the lines which bisect the angles between pairs of tangents to an Ellipse be parallel to a fixed straight line, prove that the locus of the intersection of the tangents is a rectangular Hyperbola.

102.—A point moves so that its shortest distance from a given circle is equal to its distance from a given diameter of the circle, prove that the locus of the point is a Parabola, of which the focus is at the centre of the circle.

103.—Find the locus of a point which moves so that its shortest distance from a given circle is equal to its distance from a given straight line.

104.—T is a point on the tangent at P to a Parabola, such that the perpendicular from T on FP is of constant length, show that the locus of T is a Parabola.

105.—The locus of the intersections of normals at the extremities of focal chords of a Parabola is another Parabola.

106.—The tangent at any point P of a circle meets the tangent at a fixed point A in T, and T is joined with B the extremity of the diameter passing through A, the locus of the point of intersection of AP, BT is an Ellipse.

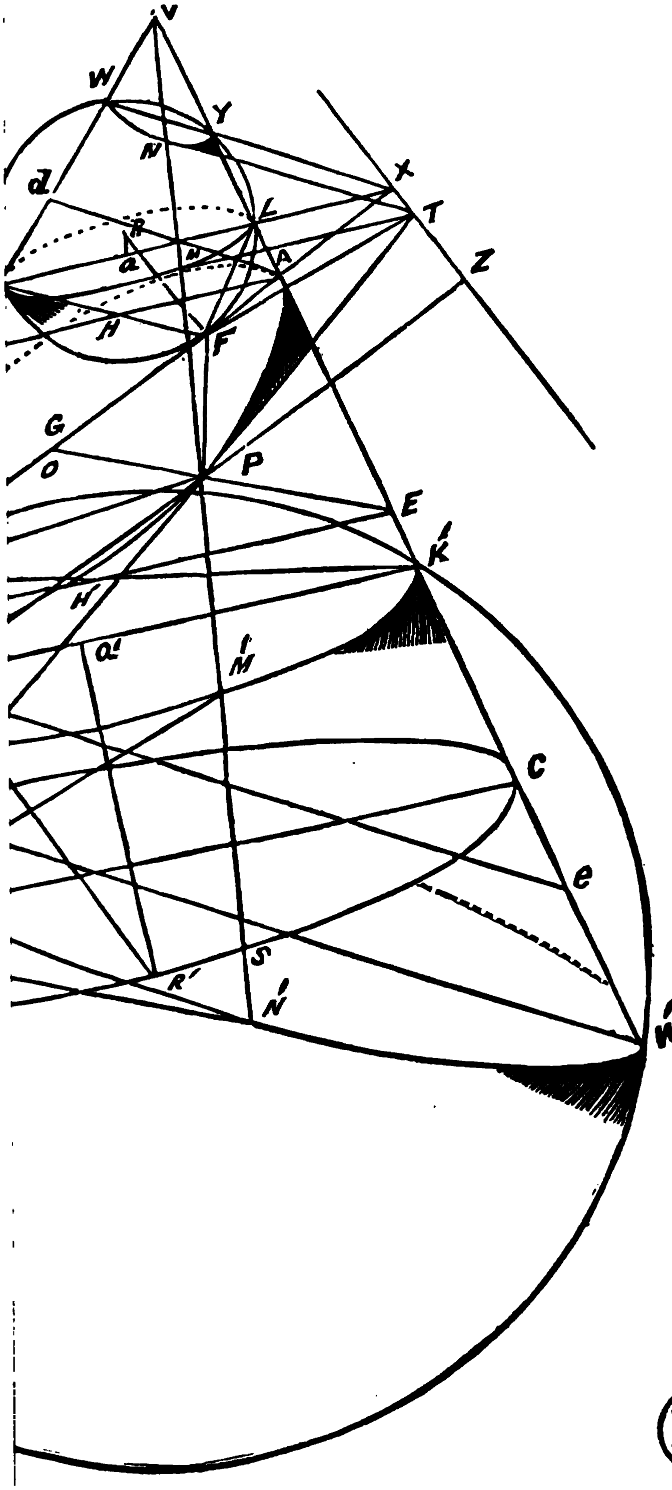
107.—If the perpendicular from the centre C of a Conic on the tangent at P meet the focal distance FP produced in R, the locus of R is a circle, the diameter of which is equal to the axis major.

108.—Find the locus of the centre of a circle touching the transverse axis, FP and FP produced where P is a point on an Ellipse.

109.—A parallelogram is described about an Ellipse, if two of its angular points lie on the directrices, the other two will lie on its auxiliary circle.

Fig. 4.

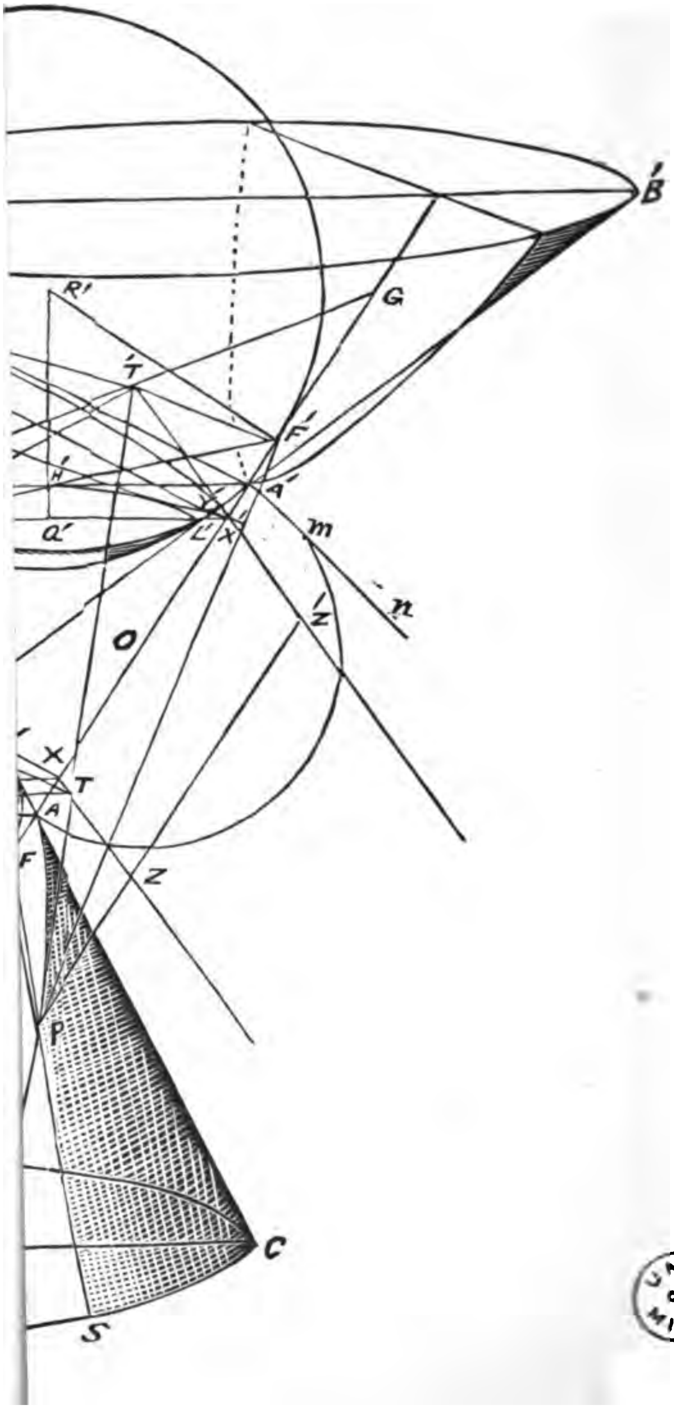
from Scalene Cone, the small Fig. showing Triangu-
n through the Vertex at right angles to Base.



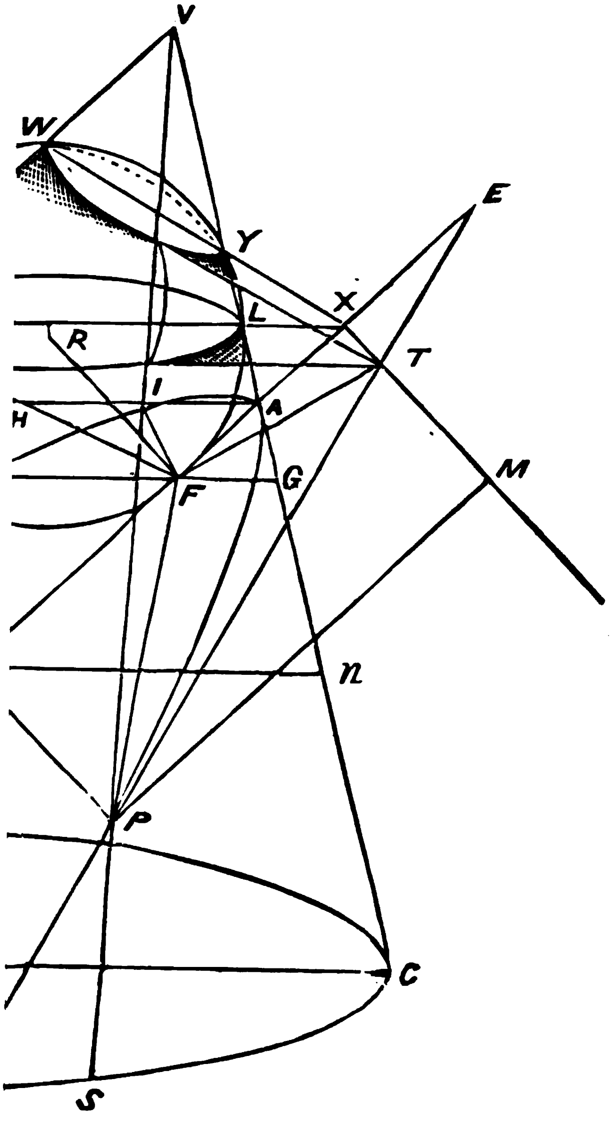
UNIV OF MIC

Fig. 5.

la from Scalene Cone, the small Fig. showing Triangular ough Vertex at right angles to Base.

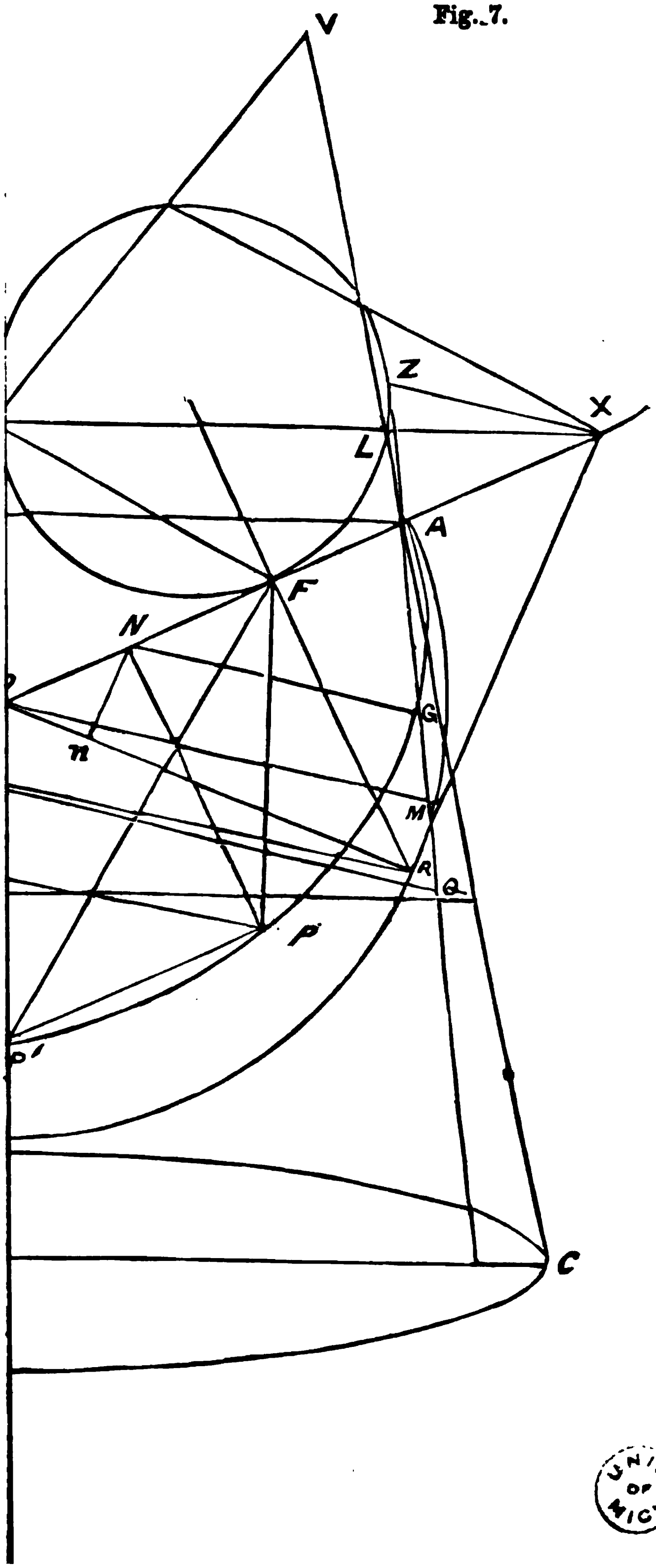


UNIV. OF MICH.



UNIV. OF MICH.

Fig. 7.



UNIV. OF MICH.

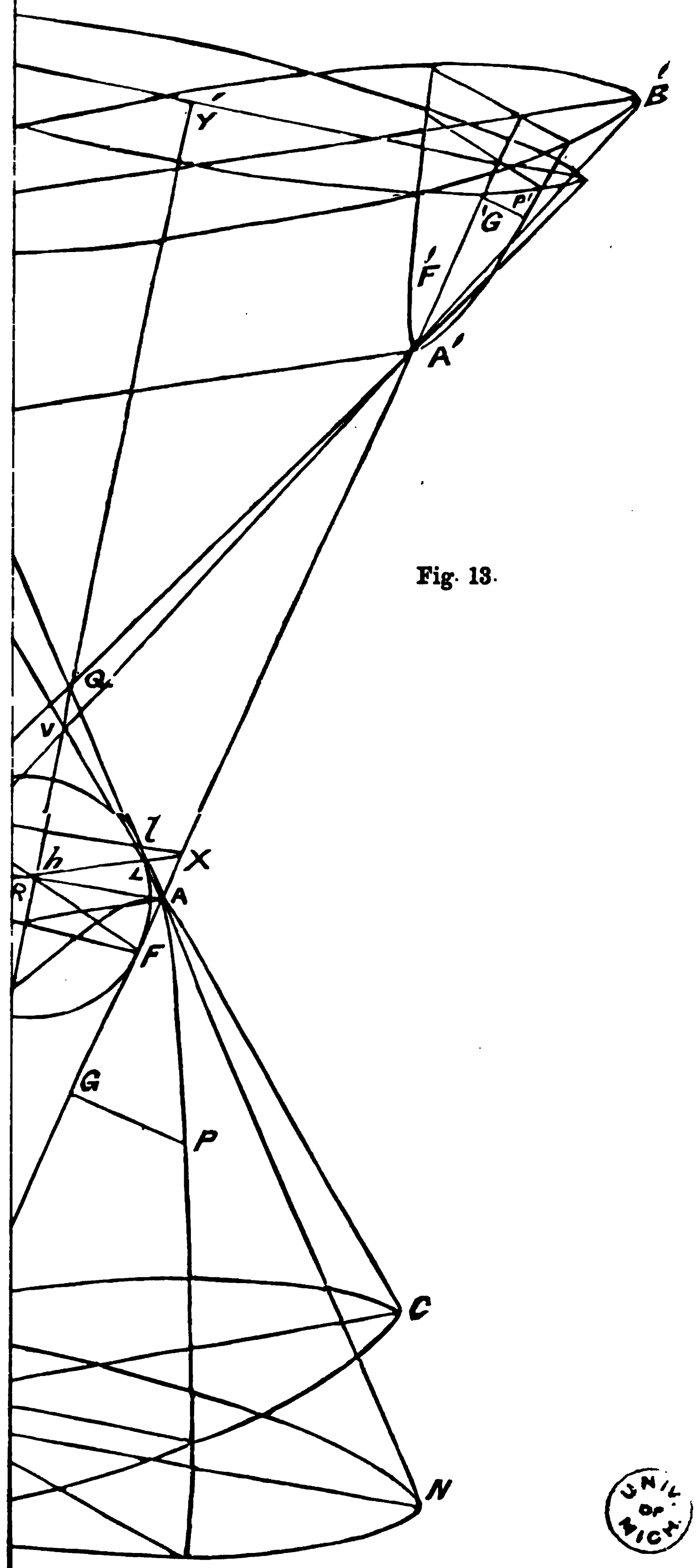


Fig. 13.

UNIV. OF MICH.

Fig. 17.

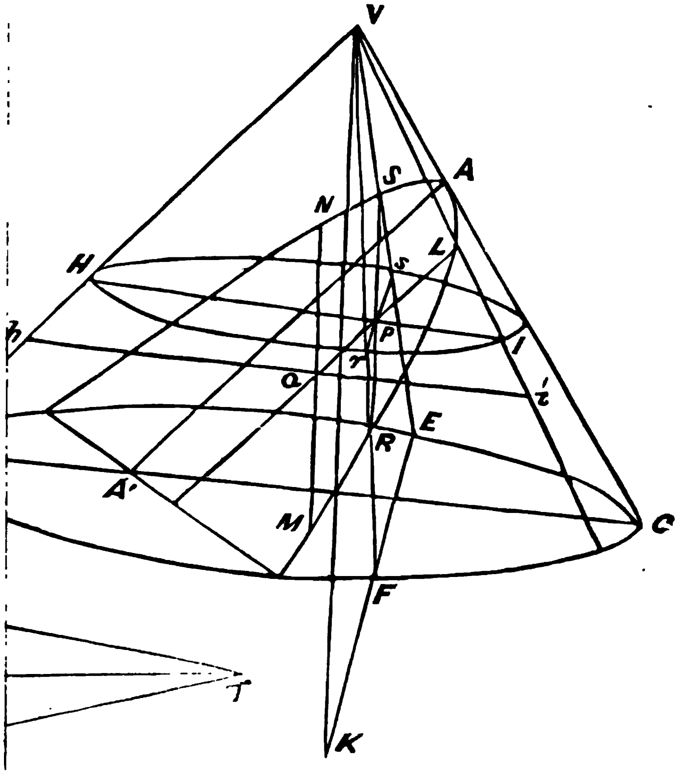
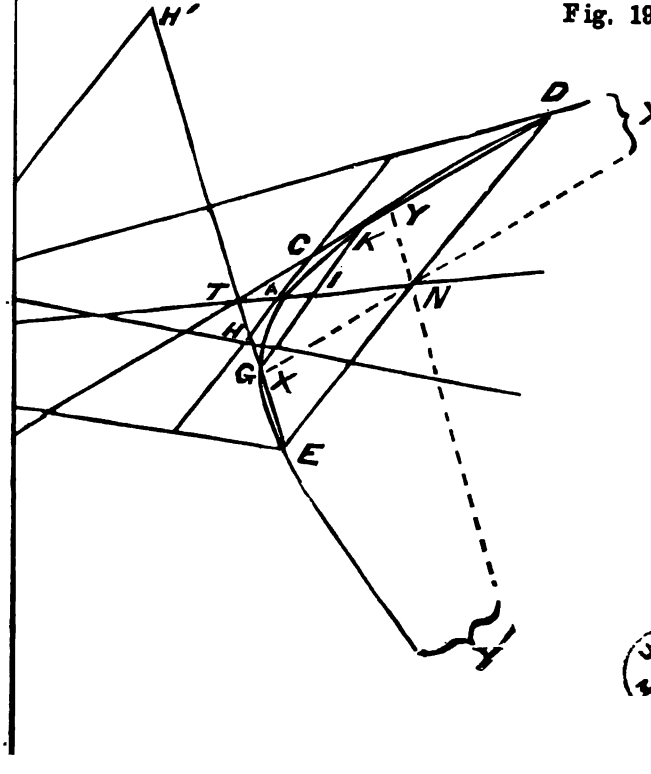


Fig. 19.



UNIV. OF

Fig. 25.

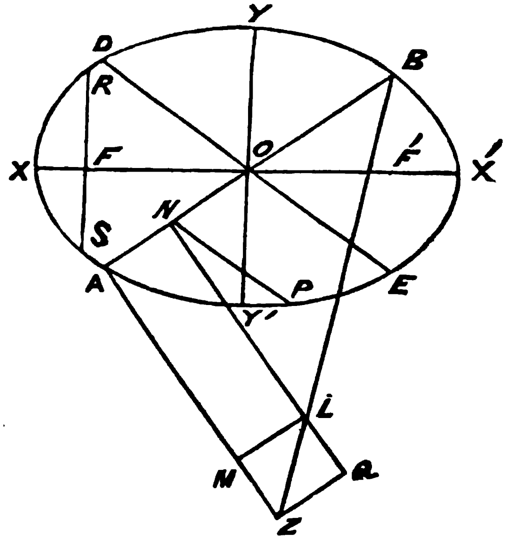
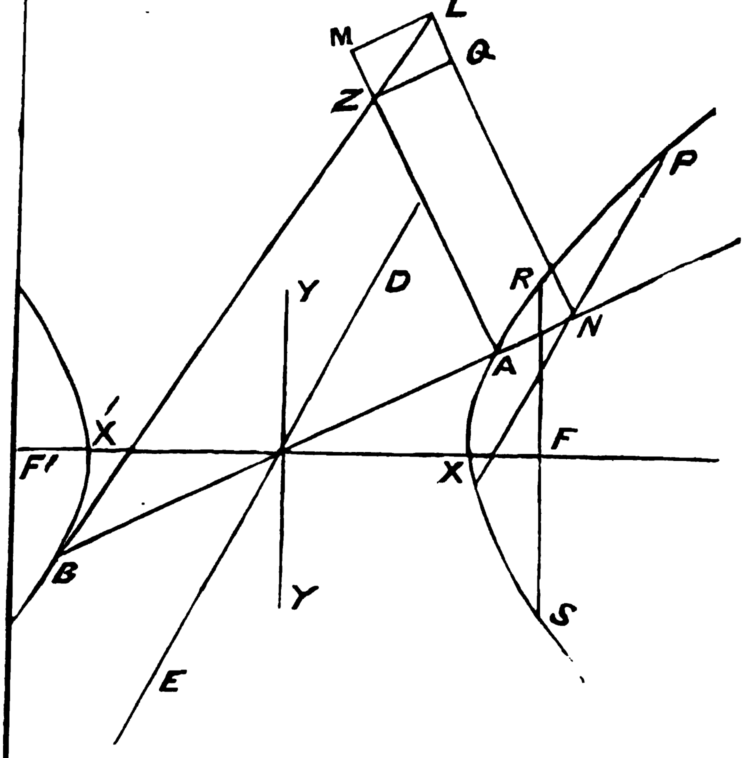


Fig. 26.



UNIV

Fig. 27.

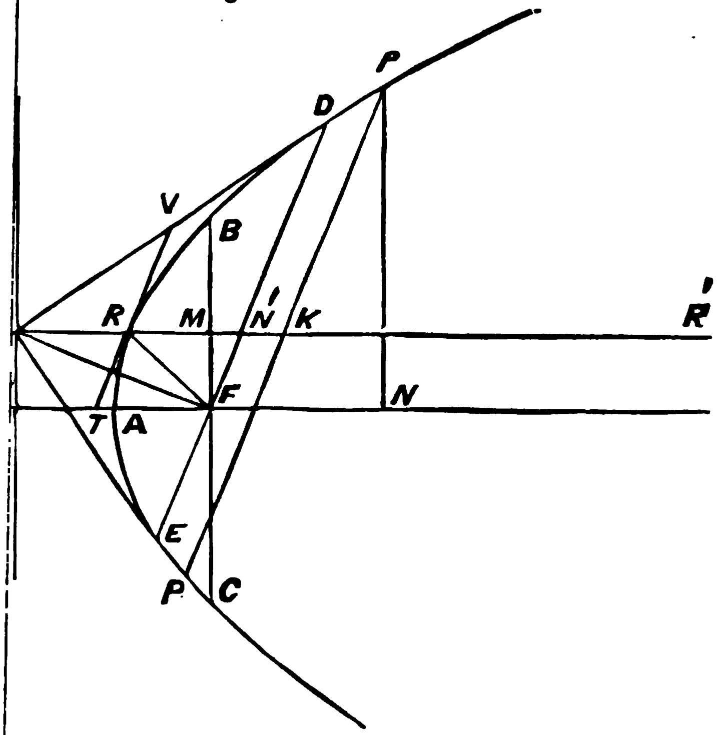


Fig. 29.

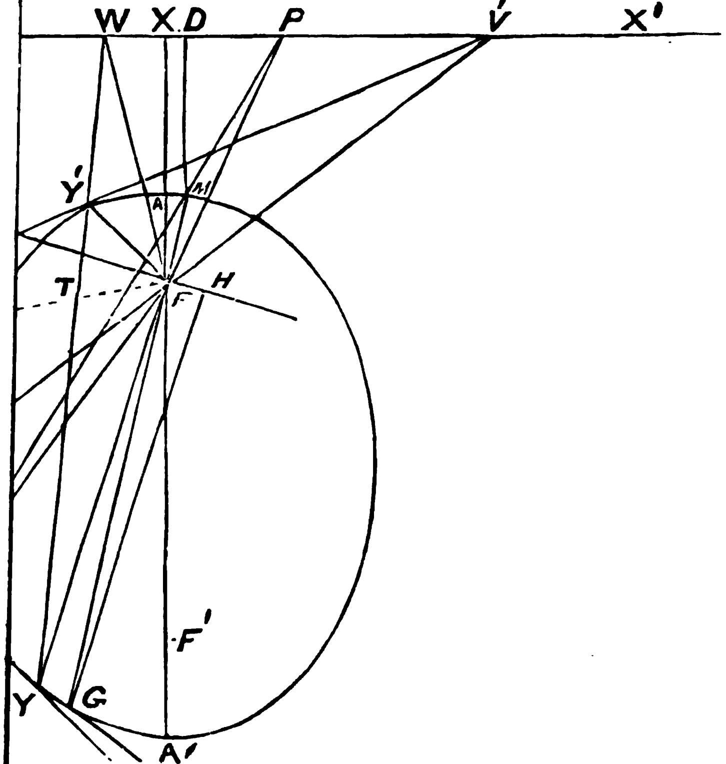


Fig. 37.

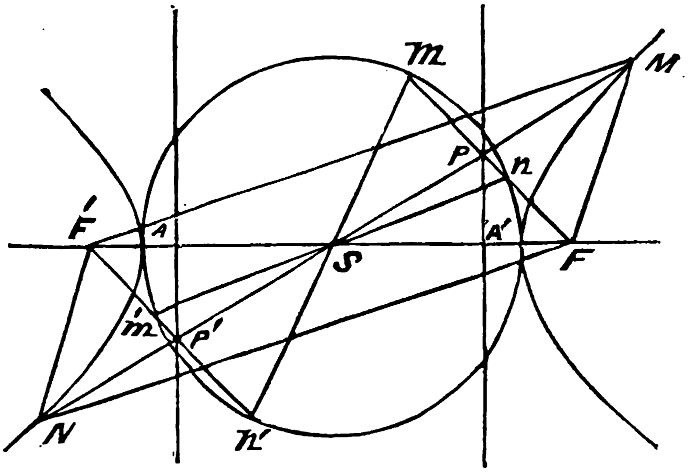
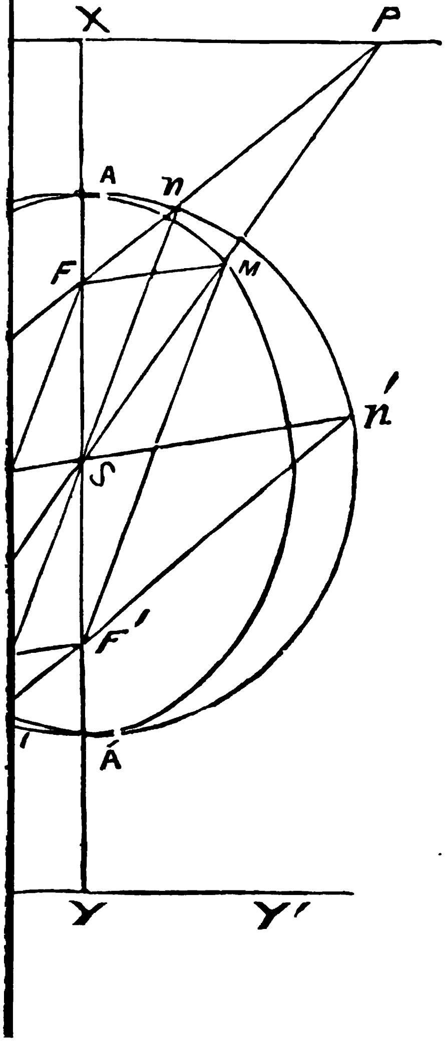


Fig. 38.



NOTE.—This paper is sent to you that you may examine it in advance of the meeting, and prepare any discussion of it which you may wish to present.

It is issued to the membership in confidence, and with the distinct understanding that it is not to be given to the press or to the public until after it has been presented at the meeting.

As there will be no supply of extra copies there, and papers are liable to be read by abstract only, preserve this copy for your use, and

BRING THIS COPY WITH YOU TO THE MEETING.

(*Subject to Revision.*)

Presented at the XIVth Meeting, New York, 1886.
American Society of Mechanical Engineers.
Advance sheets from Vol. VIII. Transactions.

CCXXX.

A NEW CONICOGRAPH.

BY ANDREW C. CAMPBELL, BRIDGEPORT, CONN.

It is with no small degree of temerity that the writer brings to the notice of this society a new device for describing the several curves known as the "conic sections." Instruments of this class, even though perfectly adapted to the purposes for which they are intended, have a very limited use in many branches of the profession, yet the problem has proven a fascinating one, and herein are presented the results of his investigation.

It is generally understood that the ellipse, parabola, and hyperbola may be drawn with such simple appliances as a string, straightedge, pencil, and a couple of pins. Ellipsographs by the score have been invented and patented in this country, to say nothing of the many devices for a like purpose which were originated abroad, so that it is not safe to rest very secure in the belief that the one herein described is entirely new, however original it may have been with the writer.

The instrument best known for describing the ellipse is the ellipse trammel, and consists of a fixed base plate with two grooves in its upper surface at right angles one with the other. A trammel bar has three adjustable points, two of which track in the grooves aforesaid, while the third traces the curve. This base plate is usually in the form of a cross having equal arms, and the smallest ellipse that can be drawn with such a device will have a major axis about twice the length of this cross, and a minor axis about one-half the major. Its application to larger ellipses is also dependent upon the size of this cross, for we cannot draw an ellipse with it whose major axis exceeds the minor by more than the length of the cross. However, this instrument has very many uses, and the principle of it is applied in almost all forms of ellipsographs. In searching the Patent-records index very many "Ellipsographs" were found, but the search was vain for parabolagraphs and hyperbolagraphs, from which it was concluded that no great success had been achieved in these directions. For describing these curves the string and straight-edge are generally resorted to, and even recommended as the only "mechanical" means, though Professor Sylvester, who investigated the "Peaucellier cell," and invented many novel uses and forms of it, developed a system of linkages from it with which it is possible to describe the ellipse, parabola and hyperbola, as well as other complicated though definite curves. Many links are brought into action in his apparatus, but the result is obtained with mathematical exactness.

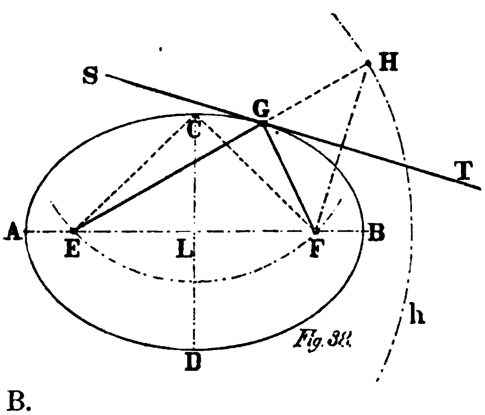
Having read the two instructive and suggestive articles in the *Scientific American Supplement*, by Professor MacCord, of Stevens Institute, in which he described two possible machines—one for drawing the hyperbola, and another for drawing the parabola—the writer became interested in the subject, and undertook to solve the same problems mechanically, and by cutting across lots the results aimed at were obtained in a comparatively simple manner, and there were embodied in a single instrument elements which admit of such transposition as to make it possible to delineate not only the parabola and hyperbola, but the ellipse as well; hence the name "*Conicograph*."

I.—The Ellipse.

In order that the "reasons why" may be well understood it might be well to show how the Gardener's ellipse is described with the string and pins, as this illustrates an important property of the ellipse, namely, that the sum of two lines drawn from the foci to meet at any point in the curve is equal to the major axis.

Having first determined upon the length of the major and minor axes, take a string (having a loop at each end) of a length equal to the major axis, and double it, using the half length thus obtained as a radius with which from one extremity of the minor axis as a center, describe an arc, cutting the major axis in two points equidistant from its center. Where said arc intersects the major axis will be the foci, and by securing the looped ends of the string aforesaid, one at each of these focal points by pins, we will find that when the string is drawn taut in any direction with a marking point, said point will lie on the curve.

Referring to the diagram, Fig. 38, let A B be the major axis and C D the minor axis of an ellipse. With A L (half of A B) as a radius and C or D as a center describe an arc cutting the line A B at the foci E and F, then C is a point in the curve, since C E + C F = A B. If the marking point be passed to the position G, and both parts of the string kept straight, the requirements are fulfilled, and G is a point in the curve, for E G + G F = A B.



Further, if the line E G be produced and G H be made equal to G F, the whole line E H will be equal to A B. This is true of course wherever the point G is taken on the curve, and the point H will always lie on the circular arc H h whose centre is E and whose radius is A B. (This line might be called the directrix of the ellipse, since its relation to the curve seems to be the same as the directrix of a parabola to the parabola.) If the angle F G H be bisected by the line S T, said line will be tangent to the ellipse at the point G and will always be at right angles to the line F H which it also bisects.

In designing a mechanism which will preserve these conditions in all positions a parallelogram of equal links (see Fig. 39) 1, 2, 3 and 4, is employed. The links 1 and 4 are jointed at the focus F; 2 and 3 to one extremity H of the adjustable radius bar 6; 3 and 4 to one end S, of the tangent bar 5, while 1 and 2 are jointed at T and control the direction of the end T of said tangent bar 5. Now if the points E and F of this linkage be located at the foci of the required ellipse whose major axis is equal to E H, the intersection of the lines E H and S T may be shown to be a point in the ellipse. At this point then is arranged a compound slide carrying a marking tool so that by swinging this system of links in the plane of the paper, about the two controlling points E and F, the circumference of the required ellipse will be traced. As constructed, this instrument will not describe the entire ellipse at one setting (owing to the interference of some of the parts), so that after drawing one-half of the figure, the points E and F must be transposed before the remaining half can be drawn. The size of the parallelogram

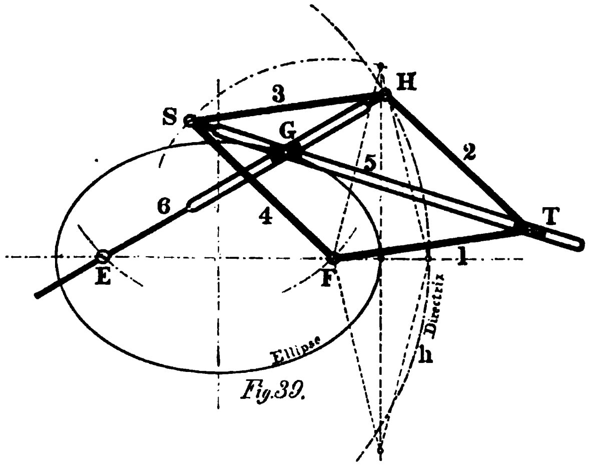


Fig. 39.

limits the maximum capacity of the instrument as an ellipsograph for describing closed curves, since the marking point G always lies *within* it. The only requisite to increase the maximum capacity is the substitution of longer links in the parallelogram. However, with the instrument shown, incomplete ellipse curves may be

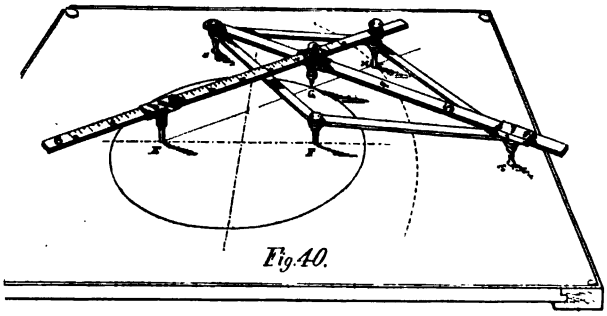


Fig. 40.

drawn with any required distance between the foci, the limit being the length of the adjustable bar (6) employed. The minimum capacity of the device is also limited, the consideration in this case being the least possible distance obtainable between the focus F and the marking point, when the later is at that extremity of the major axis which is nearest the focus F aforesaid. See dotted position in Fig. 39.

In constructing the instrument shown in perspective in Fig. 40, the links of the parallelogram should be of some light stiff material, preferably steel, and substantially jointed at their extremities to the upper end of four short supporting pillars, those at S H and T terminating at the base in the form of smooth-bottomed disks which may slide freely on the plane upon which the ellipse is to be traced, the fourth pillar at F having a shouldered needle-point in its lower end which punctures the drawing paper in one of the foci of the required ellipse, and forms one of the controlling points of the mechanism.

The tangent bar 5, is jointed at S to the upper end of one of these supporting columns, and extends toward the opposite angle of the parallelogram, where it passes freely through a slide at the top of the column T, so that whatever be the distance between the points F and H, the tangent bar exactly bisects it. The adjustable link 6, lies in a plane above the tangent bar 5, and one end thereof is jointed to the upper extremity of the column at H. Upon this bar is placed an adjustable slide which is secured in position by a clamping screw entering it from below, said screw forming the supporting column at E, which terminates at the base in the form of a shouldered needle-point. This slide must be so placed along the bar 6, that the distance between the centers of the columns E and H shall be equal to the major axis of the ellipse to be drawn. By arranging a scale on said bar, the only setting required can be very readily made.

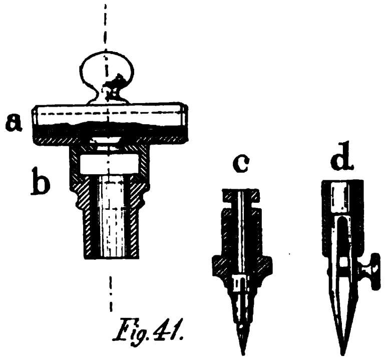


Fig. 41.

The bars 5 and 6 are made of brass tubing in preference to other material, and the general stability of the apparatus might be increased by substituting a standard at E with a large flat base, and swinging the bar 6 from its upper end, thus preventing the twisting action which otherwise tends to deflect said bar.

The compound slide at G, which carries the marking point is shown enlarged in Fig. 41, and, as will be seen, consists of the two parts **a** and **b** of rectangular cross-section through the first of which

UNIV. OF MICH.

slides the adjustable bar 6, while through the latter slides, the tangent bar 5, and these two parts are so jointed to one another that they may revolve in parallel planes about a common vertical axis. As the metal tubing from which these parts are made is quite thin, the construction shown was resorted to in order to get a swivel between them—using a disk counter sunk into the upper slide and riveted into the lower one. Extending downward from the under side of the lower, or tangent bar slide, is a tubular socket, into which may be placed the marking device, either a pencil **c** or a pen **d**, and since the former is subject to wear, it is arranged in a spring socket as shown, thus also insuring a uniform pressure over slightly irregular surfaces.

Your attention is particularly called to two important advantages possessed by this instrument. First, when the blades of the pen are placed in proper position to draw a line in the direction of the tangent bar, said pen is always guided thereby exactly in the direction of the curve, and thus gives a better line than can be drawn by any other instrument made for a like purpose. As far as the writer can discover, this is the first time this result has been obtained in an ellipsograph. The second feature is that it is possible to draw any number of curves parallel to the ellipse, either larger or smaller than the original figure, and this is often very desirable, as, for instance, in drawing elliptical gearing. A curve parallel to an ellipse is *not* an ellipse, and is therefore out of reach of all ellipsographs heretofore made. To do this with the instrument shown, it is only necessary to place the marking point the required distance from the center of the compound slide and at right angles to the direction of the tangent bar. The attachment shown in Fig. 42 is designed to be used in place of the regular marking device, and can be readily secured in proper position on the pendant sleeve of the compound slide, and easily adjusted to draw curves parallel to the ellipse.

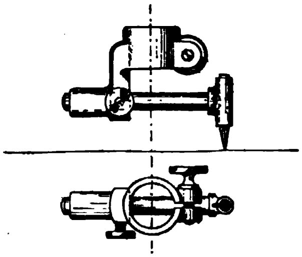


Fig. 42.

II.—The Parabola.

In reference to the particular property of the parabola, of which advantage is taken, Davies' *Descriptive Geometry* may be quoted:

"If a right line, E D, and a point, F (Fig. 43), be taken in the plane of the paper, and a point, as G, be so moved in this plane that the distance from F be constantly equal to its distance from E D, that is, G F = G H, the point G will describe a curve called a *parabola*. The line E D is called the *directrix* of the parabola, and the point F the focus; the line A F, perpendicular to E D, the *axis*, and the point B, in which the axis intersects the curve, the *vertex of the parabola*. Points of the curve may be found thus: take any point in the directrix, as E, and draw E F to the focus; draw, also, E L perpendicular to the directrix, and at the point F make the angle E F L equal to the angle F E L, then the point L, at which the lines E L and F L intersect, is a point of the curve.”

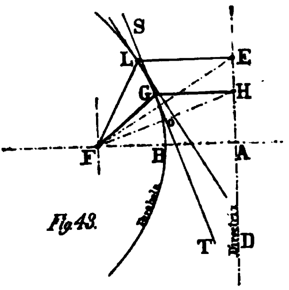


Fig. 43.

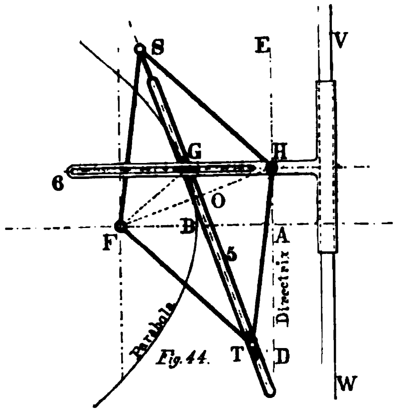


Fig. 44.

How nearly this Conicograph can be made to fulfill these conditions will be seen by the following: reference being made to the diagram, Fig. 44, and the perspective, Fig. 45, wherein the bar 6 is shown as controlled by its “Tee” head and the straight-edge V W, so that it lies constantly parallel to the axis of the parabola, and the point H thereon may be made to lie always on the directrix E D. Let F be located in the focus, and the angle of the parallelogram opposite to F be pivoted to the point H of the bar 6, then F H is exactly bisected at O by the tangent bar S T, since

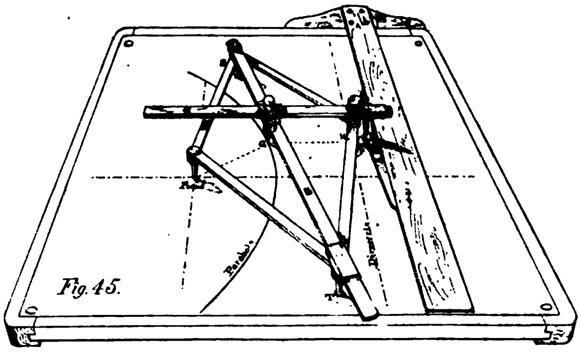


Fig. 45.

UNIV. OF MICH.

by construction S H = S F and H T = F T. If, from the point H, G H be drawn perpendicular to the directrix, it will intersect the tangent bar S T at G, a point in the curve for G H = G F and H O = O F. Since G O is common to the two triangles, O G H and O G F, the angles F H G and H F G are equal.

In order to adapt this instrument to these conditions, the adjustable slide E shown in Figs. 39 and 40 must be removed from the bar 6, and the attachment shown in Fig. 46 secured to said bar by the screw at H, which transforms the bar 6 into the "blade" of a Tee square, the attachment forming the "head." In drawing the parabola with this device a straight-edge must be placed on the paper, parallel to the directrix, and a little way from it. Along this straight-edge the head of the Tee square must be moved while describing the curve, thereby keeping G H at right angles, with the directrix E D and H always on it.

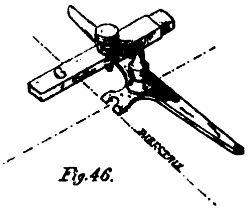


Fig. 46.

The attachment shown in Fig. 42 is applicable to the parabola as well as to the ellipse, and with it curves may be drawn parallel to the parabola. In inking these parallel curves the pen is always directed by the tangent bar, as in the case of the ellipse, and with equally perfect results.

III.—The Hyperbola.

In reference to the hyperbola Davies may be again quoted:

"If two points, E and F (Fig. 47), be taken in the plane of the paper, and a point, G, be moved with the condition that it continue in the plane of the paper, and that the difference between the distance F G and E G be a constant quantity, the point G will describe a curve, P A G, called an hyperbola. . . . The points E and F are called the *foci*, the line B A is named the *transverse axis*, and the points B and A, in which it intersects the curve, are the vertices of the hyperbolas."

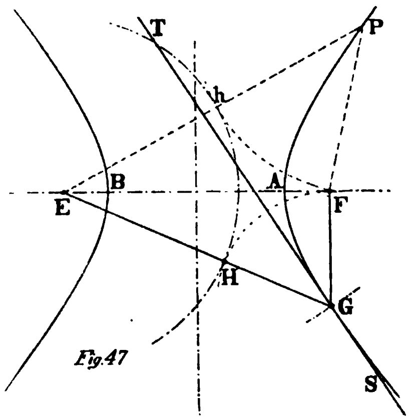


Fig. 47

Applying this Conicograph to describe an hyperbola, let the needle-point of F (See Fig. 48, and perspective, Fig. 49), be located at the focus of the hyperbola to be drawn, and the needle-point of E at the focus of the opposite hyperbola.

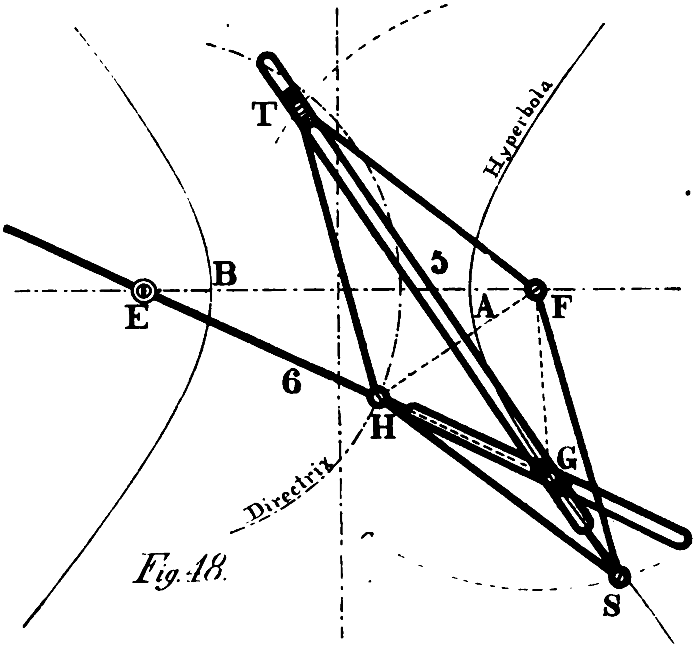


Fig. 48.

In drawing an ellipse the slide G, carrying the marker, is always between the points E and H, while for drawing the hyperbola it must always be placed beyond H. In order to have it so located, the point H, in the adjustable bar 6, must be pivoted (by means of the removable screw at that point) to that angle of the parallelogram opposite the focus F, one end of

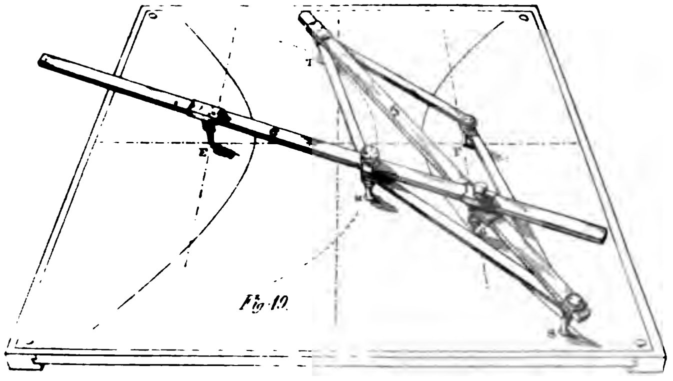


Fig. 49.

UNIV. OF MICH.

said bar extending through the compound slide G, the other through the adjustable slide E. Taking G as a point on the curve, G F = G H and E G — G F = E H, which latter is a constant, therefore, if E be taken as a center and E H as a radius, the point H will describe the curved directrix H h (Fig. 47). Bisecting F H with a line at right angles thereto will locate the tangent S T. The angle G F H being equal to F H G, locates G as a point on the curve, and also as the point of tangency of the tangent bar 5 to the curve. Further, E P — F P = E h = E H, therefore P is another point on the curve of the hyperbola.

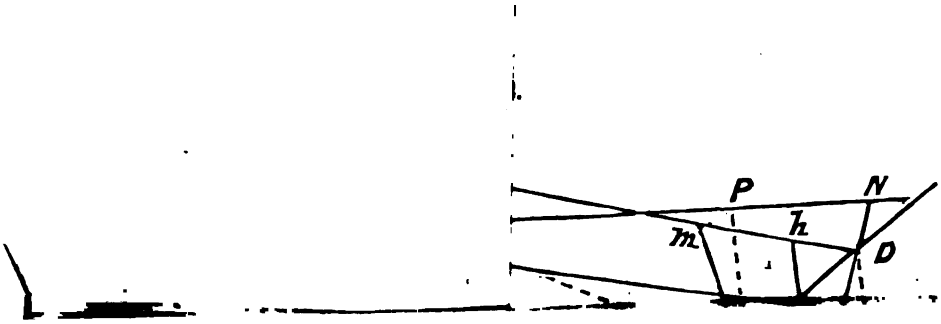
In drawing an hyperbola, then, with this Conicograph, the "settings" are as follows:

First, locate E on the adjustable bar 6, at a distance from H equal to the transverse axis B A of the hyperbola—then make A F = B E, and locate the needle-points of the instrument in the foci E and F.

By swinging the point H of the linkage as far as possible above and below the axial line, the marker attached to the compound slide at G will trace the required curve. To draw the curve of the opposite hyperbola the points E and F must be transposed, as in the case of the ellipse.

The attachment shown in Fig. 42 may be used for drawing curves parallel to a given hyperbola.

This drawing-instrument is presented for your consideration more as a novelty and curiosity than as a perfect machine, though even in its imperfect state it proves to be a very serviceable Conicograph, as the few specimens of drawings shown at the meeting in connection with this paper will attest.



UNIV. OF MICH.